## Group theoretic formulation of complementarity

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Bohr's complementarity principle is a defining feature of quantum physics [1]. In essence it represents the dichotomy between the particle and wave nature of objects; the particle properties are typically symbolized by well-defined position and the wave properties by well-defined momentum. The related study of the simultaneous measurement of non-commuting observables also has a long history [2].

Here we generalize the complementarity principle as follows. We note that waves are essentially invariant to spacial translations whereas particles are essentially localized in space. In other words, waves are symmetric and particles are asymmetric with respect to the translation group. We generalize this notion of associating wave and particle nature with symmetry and asymmetry to an arbitrary finite symmetry group G. We consider the generalized particle and wave nature of a system with state density operator  $\rho$ . For convenience we call the generalized particles simply "particles", and similarly generalized waves "waves", and we refer to transformations by the group as "translations". Let G have the unitary representation  $T_g$  for  $g \in G$  on the system's Hilbert space, and let the order of G be |G|.

We first consider particle nature. Particle-like states are localized with respect to the group  $G = \{g\}$  and hence their translation  $\rho \mapsto T_g \rho T_g^{\dagger}$  can carry information. We imagine an information theoretic scenario between two separated parties A and B as follows. Party A sends the system in the translated state  $\rho_g = T_g \rho T_g^{\dagger}$  for  $g \in G$  with probability 1/|G| to B, and B wishes to make measurements to determine the value of g. We define the particle nature  $N_P(\rho)$  of  $\rho$  as the maximum of the mutual information between A and B over all possible measurements at B. The maximum is given by the Holevo bound [3] as

$$N_P(\rho) \le S(\mathcal{G}[\rho]) - S(\rho) = A_G(\rho) \tag{1}$$

where  $S(\rho)$  is the von Neumann entropy of  $\rho$  and  $\mathcal{G}[\rho] \equiv \sum_g T_g \rho T_g^{\dagger}/|G|$  is the average state received by B. The quantity  $A_G(\rho)$  is the asymmetry of  $\rho$  with respect to the group G [4]; hence the particle nature is bounded by the asymmetry of  $\rho$  which is consistent with our identification of particles with asymmetry.

We now consider the analogous scenario for the wave nature. The wave properties of the state are invariant to translations  $T_g$  for  $g \in G$ , and so in terms of the wave nature the state  $\rho$  is equivalent to  $T_g \rho T_g^{\dagger}$  and also to  $\mathcal{G}[\rho]$ . We imagine that party A encodes information in the wave nature of the state, using a suitably restricted class of operations, and sends the

system to B who then decodes the information using measurements. We define the wave nature  $N_W(\rho)$  as the maximum of the mutual information between A and B over all possible measurements at B. We eventually find that

 $N_W(\rho) \leq \ln(D) - S(\mathcal{G}[\rho]) = W_G(\rho)$  (2) where D is the dimension of the system's state space. The quantity  $W_G(\rho)$  is the extractable work from a thermal reservoir using the state  $\mathcal{G}[\rho]$ , i.e. it is a logarithmetic measure of the purity of the state  $\mathcal{G}[\rho]$  [4]. The more invariant the state  $\rho$  is with respect to translations  $T_g$  of G, the more pure  $\mathcal{G}[\rho]$  will be, and so  $W_G(\rho)$  is also a measure of the symmetry of  $\rho$  with respect to G [4]. Thus the wave nature is bounded by the symmetry of  $\rho$  which is consistent with our association of waves with symmetry.

Combining these two expressions yields the complementarity relation:

$$N_P(\rho) + N_W(\rho) \le \ln(D) . \tag{3}$$

That is, the sum of the particle and wave nature is bounded by the maximum information that can be carried by the system.

The association of particle nature with asymmetry and wave nature with symmetry can be interchanged by changing the group translation. For example, in the momentum representation a wave-like state is highly localized and so it is asymmetric to momentum boosts (i.e. momentum translations), whereas a particle-like state is delocalized in the momentum representation and as such it is invariant to such boosts. We will discuss the implications of this feature. Also we re-examine previous work (e.g. Englert's fringe visibility thought experiment [5]) in terms of our approach. Our formalism also allows the study of the simultaneous measurement of complementary properties in an information theoretic framework

## References

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