

# Comment on “One-dimensional nonlinear steady infiltration”

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*Basha* [1999] has developed an interesting approach for steady infiltration when the soil-water conductivity obeys the relation

$$K = \frac{1}{1 + \psi^n}. \quad (1)$$

In this comment, Basha's notations are followed;  $K$  is a reduced conductivity so that  $K = 1$  for  $\psi = 0$ , and the matric potential is also made dimensionless using some characteristic potential. Note that as a result,  $\psi > 0$  for unsaturated soil. In general, the governing equation is

$$K(d\psi/dz + 1) = Q, \quad (2)$$

where  $Q$  is the dimensionless flux and  $z$ , positive downward, is a dimensionless distance. To integrate (2), an additional boundary condition is used:

$$\psi = \psi_b, \quad z = z_b. \quad (3)$$

*Basha* [1999] then uses an ingenious expansion to solve the problem, assuming  $n$  to be sufficiently large.

*Basha's* [1999, equation (5)] approximation yields

$$\psi = 1 - \frac{1}{n-1} \ln(\alpha) + \frac{1}{(n-1)^2} [\ln(\alpha) + \frac{1}{2} \ln^2(\alpha)], \quad (4)$$

when  $z_b \rightarrow \infty$ , whereas the exact solution is [see *Basha*, 1999, equation (52)]

$$\Psi = (1/\alpha)^{1/n}, \quad (5)$$

with

$$\alpha = Q/(1 - Q). \quad (6)$$

*Basha* adds that his “Equation (51) compares well with the exact limit . . . especially for relatively high  $n$  values. The relative error ranges from 5% for  $n = 3$  to less than 1% for higher  $n$  values.” This cannot be true for all values of  $\alpha$ . For instance, for a given  $n$  if  $\alpha \gg 1$ , that is,  $Q \approx 1$ ,  $\psi \approx 0$  (see (5)), whereas (4) gives  $\psi \gg 1$ . Although less striking, the error is also large for  $\alpha \ll 1$ . As an illustration, if we write  $\alpha = \exp(-\lambda n)$ , then (5) yields  $\psi = \exp(\lambda)$ , whereas (4) gives for  $n \gg 1$ ,  $\psi = 1 + \lambda + \lambda^2/2$ , that is, the beginning of the expansion of  $\exp \lambda$ , so that the result is good for small  $\lambda$  only.

It is clear that it would be preferable if the approximate solution reduced to (5) when  $z_b \gg 1$ . This is satisfied in the following. To do so, (2) is approximated by replacing  $K/(K - Q) = 1 + Q/(K - Q)$  using

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$$K - Q \approx C \exp[-\alpha(z_b - z)]. \quad (7)$$

This type of approximation has been used in the past and becomes exact when  $K$  is of the form prescribed by *Gardner* [1958] and it was used in the present context by *Warrick* [1974]. The same approach was also used by *Parlange* [1982] in a drainage problem. Here (7) is only an approximation, and  $C$  and  $\alpha$  are two positive constants to be determined later. We note in passing that as  $z_b \rightarrow \infty$ , (7) yields  $K = Q$ , that is, the exact solution. Plugging the approximation into (2) yields by integration

$$(Q + C) \exp[-\alpha(\psi - \psi_b)] - Q = C \exp[-\alpha(z_b - z)], \quad (8)$$

which is identical to the approximation of (7) when  $K = (Q + C) \exp[-\alpha(\psi - \psi_b)]$ , that is, a Gardner-type behavior. For a different  $K$ , (8) should usually be more accurate than (7) since it results from one iteration. Here  $\psi_b$  is defined as the value of  $\psi$  where  $z = z_b$ , and  $C = K_b - Q$ , where  $K_b$  is the value of  $K$  at  $\psi = \psi_b$ .

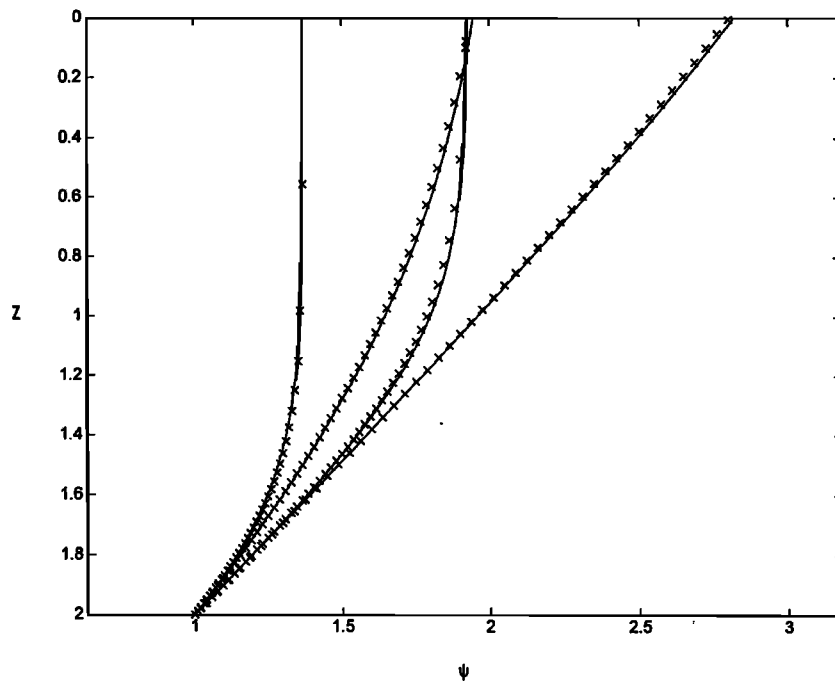
We are now going to apply the above results to the two examples discussed by *Basha*. The first example follows the cases in Figure 1 of *Basha*, that is,  $n = 3$  and 7 and  $Q = 0.1$  and 0.01, with  $\psi_b = 1$  and  $z_b = 2$ . Thus we know everything in (8) except  $\alpha$  which has to be obtained judiciously. It is important to reiterate that this being a “comment” on *Basha's* work, the determination of  $\alpha$  pertains to his examples. A more general theory based on the present approach will be published later. In particular, we are limiting ourselves to the case where  $K$  obeys (1) exactly. Hence  $K_b = 1/2$ . Equation (8) now becomes

$$\frac{1}{2} \exp[-\alpha(\psi - 1)] = Q + (\frac{1}{2} - Q) \exp[-\alpha(2 - z)]. \quad (9)$$

To find  $\alpha$ , (9) can be forced to satisfy (1) and (2) at some point. It already satisfies  $\psi = 1$  at  $z = 2$ . An obvious choice is the point  $z = 0$ , where  $\psi = \psi_0$ . Then

$$(\frac{1}{2} - Q) \exp(-2\alpha) + Q = 1/(1 + \psi_0^n) = \frac{1}{2} \exp(-\alpha(\psi_0 - 1)), \quad (10)$$

which provides two equations and thus yields  $\alpha$  (and  $\psi_0$ ). It is easy to solve (10) by iteration. Starting with  $\alpha = n/2$  (the value obtained by crudely fitting (9) near  $z = 2$ ), the left-hand side of (10) is used to obtain  $\psi_0$ . The right-hand side of (10) then provides a new  $\alpha$ . Convergence is rapid, and the calculation easy. Figure 1 then repeats the four cases of *Basha's* Figure 1. Obviously, the results are very good by comparison with the exact solution of *Basha*. The accuracy of the present approximation is similar to that of *Basha's*, but the analytical form of (9) is neatly simpler than *Basha's* approximation.



**Figure 1.** Pressure distribution obtained from the exact solution (crosses), as given by *Basha* [1999], and the present approximation (solid curve) as given by equation (9). The four cases from the left to the right correspond to  $n = 7$ ,  $Q = 0.1$ ;  $n = 3$ ,  $Q = 0.1$ ;  $n = 7$ ,  $Q = 0.01$ ; and  $n = 3$ ,  $Q = 0.01$ . The values of  $\alpha$  as given by equation (10) are  $\alpha = 4.373$ ,  $1.513$ ,  $4.215$ , and  $1.352$ , respectively.

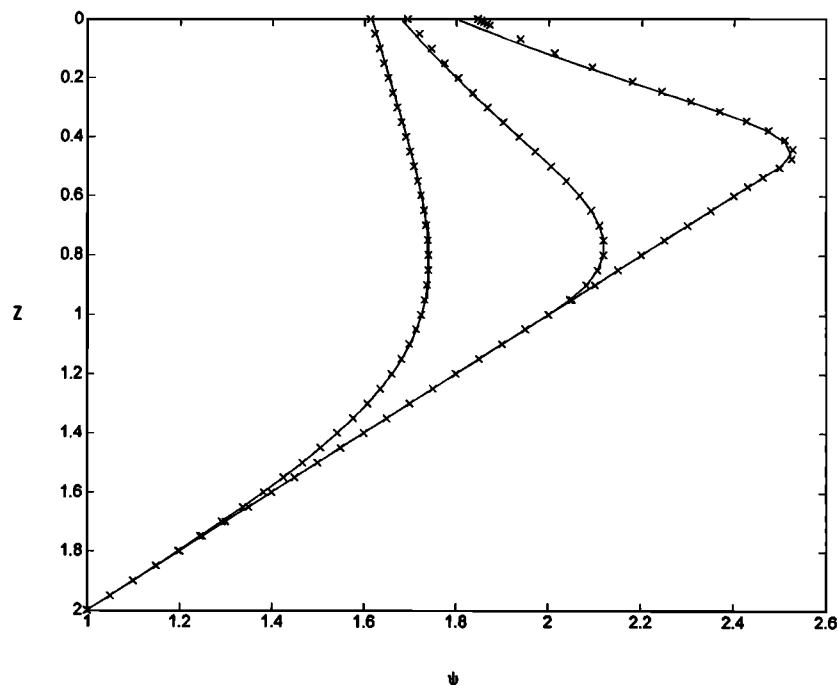
The second case considered by Basha is with root uptake for  $z < D$ , so that  $Q$  in the right-hand side of (2) is replaced by  $Q - \bar{F}$ , with

$$\bar{F} = Qz/D, \quad z \leq D. \quad (11)$$

For  $2 > z > D$ , (9) applies for  $Q = 0$  or, exactly,

$$\psi - 1 = 2 - z, \quad z \geq D. \quad (12)$$

A similar result to (9) is now obtained for  $z < D$  or, since  $\psi = 3 - D$  at  $z = D$  from (12),



**Figure 2.** Pressure distribution for  $n = 5$ ,  $Q = 0.1$  corresponding to three rooting depths,  $D = 2$ ,  $D = 1$ , and  $D = 0.5$ , from left to right. The present approximation (solid curve) is given by equation (12) for  $z \geq D$  and by equation (15) for  $z \leq D$ . Values of  $\alpha$  given by equation (16) are  $\alpha = 2.888$ ,  $2.364$ , and  $1.971$ , respectively. The numerical solution (crosses) is the same as given by *Basha* [1999].

$$K_D \exp[-\alpha(\psi - \psi_b)] - Q = (K_D - Q + A) \exp[-\alpha(D - z)], \quad (13)$$

where  $K_D = 1/[1 + (3 - D)^n]$  and

$$A = -\alpha \int_z^D \bar{F}(x) \exp[\alpha(D - x)] dx, \quad (14)$$

and from (11),

$$\frac{1}{1 + (3 - D)^n} \exp[-\alpha(\psi - 3 + D)] = \frac{1}{1 + (3 - D)^n} \exp[-\alpha(D - z)] + Q \left(1 - \frac{z}{D} - \frac{1}{\alpha D}\right) + \frac{Q}{\alpha D} \exp[-\alpha(D - z)]. \quad (15)$$

Once again  $\alpha$  has to be obtained. In this case it is easy to match (15) when  $d\psi/dz = 0$  at  $\psi = \psi^*$  or

$$\frac{Q}{\alpha D} \ln \left[ 1 + \frac{\alpha D/Q}{1 + (3 - D)^n} \right] = \frac{1}{1 + \psi^{*n}} = \frac{1}{1 + (3 - D)^n} \cdot \exp[-\alpha(\psi^* + D - 3)]. \quad (16)$$

As with (10), we can start with  $\alpha = n/2$  to calculate the left side of (16). Then the midterm gives an estimate of  $\psi^*$ , and the right side gives a new value of  $\alpha$  to be used for reiteration. Rapid convergence results. Figure 2 shows the results for  $D =$

2, 1, 0.5;  $n = 5$ ; and  $Q = 0.1$ , reproducing Basha's Figure 2. Again the approximation of (15) is remarkably accurate by comparison with a numerical solution. For this example the analytical approximation is not only simpler than Basha's but, in addition, neatly more accurate.

In conclusion, using a very simple exponential approximation, we were able to provide simple and accurate results for Basha's examples. A general theory will be given later.

## References

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