

Chapter 12

A NOTE ON THE PROPAGATION OF WATER TABLE WAVES: DUAL LENGTH SCALE CONSIDERATIONS

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Abstract

The problem of a coastal aquifers forced by oscillations in an adjacent sea and/or estuary across a sloping boundary has recently received considerable theoretical attention. Despite such a wealth of mathematical advancements, stringent testing of the limitations of these models has yet to be undertaken. In all of the currently available analytical solutions it has been assumed that a single length scale is sufficient to account for both the amplitude decay rate and the rate of increase in phase lag (the wave speed) as the water table wave propagates landward. All of the available field and laboratory data however indicate that this is not the case. That is, the real part of the water table wave number (the amplitude decay rate) is not equal to the imaginary part (the rate of increase in the phase lag). In this chapter, the detailed laboratory measurements of *Cartwright et al.* [2004] are used to highlight the limitation of assuming a single length scale in these mathematical models. In a step towards overcoming this limitation, a new approximate analytical solution is derived which allows for two different length scales as observed in the available data. In the absence of the ability to accurately predict the water table wave number using basic aquifer parameters, all of the solutions are applied to the data using water table wave numbers estimated from the data. Accurately predicting the water table wave number based on measurable aquifer parameters remains a challenge.

Keywords: water table waves; sloping boundary; finite depth aquifer; capillary fringe; seepage face; amplitude decay, phase lag.

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1. Introduction

Coastal aquifers are dynamic in response to forcing from adjacent clear water bodies such as oceans, estuaries and rivers. Understanding and accurate prediction of these dynamics is an important pre-requisite for furthering the understanding of processes in other disciplines such as coastal erosion, coastal ecology and coastal water resources. The dynamic nature of coastal aquifers will affect the mixing of salty, oxygen-rich seawater with fresh, oxygen-depleted groundwater [e.g. *Robinson and Li, 2004*] and has also been linked to sediment mobility on beaches [e.g. *Elfrink and Baldock, 2002*].

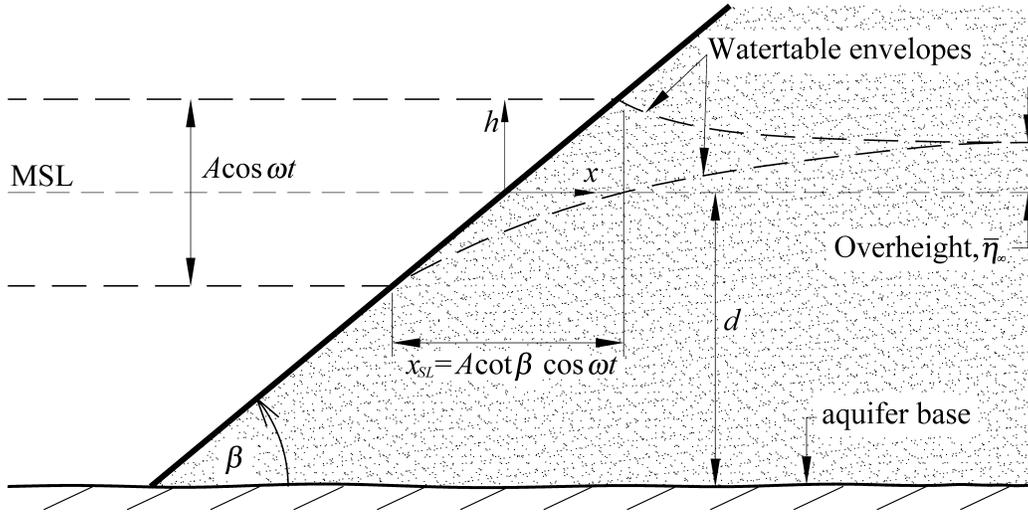


Figure 1. Schematic of the sloping boundary condition.

Generally, the interface between an ocean and aquifer is non-vertical, giving an interesting mathematical problem due to the moving shoreline coordinate [cf. Figure 1]. To date, an exact analytical solution to the sloping boundary problem is yet to be derived with all available solutions derived using the perturbation approach [cf. *Nielsen, 1990; Li et al., 2000a; Teo et al., 2003*]. In all of the existing analytical solutions, the assumption of a single length scale to account for both amplitude decay and phase shift has been made. That is, the amplitude decay rate is the same as the rate of increase in phase lag as the water table wave propagates in the aquifer.

However, data from both laboratory and field observations clearly show that this is not the case. For example, Figure 2 provides a compilation of the available field data on the propagation and decay of tidally driven water table waves and clearly shows that the amplitude decay rate is greater than the rate of increase in phase lag ($k_r > k_i$). Similar observations have been made in the laboratory [e.g. *Nielsen et al., 1997; Cartwright et al., 2004*]. This duality of length scales has been shown previously to be due to either capillarity [cf. *Barry et al., 1996*], vertical flow effects (finite depth aquifers) [cf. *Nielsen et al., 1997*] or a combination of both [cf. *Li et al., 2000b*]. The $k_r = k_i$ line shown in Figure 2 is the equivalent to assuming a single length scale and the comparison with the data highlights the significant limitation of such an assumption.

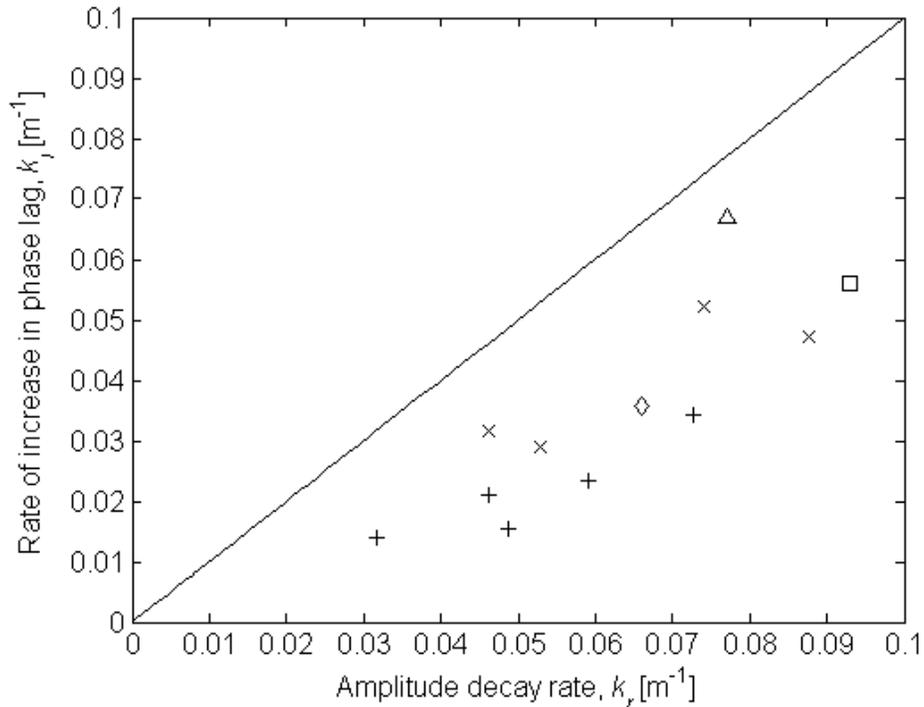


Figure 2. Compilation of the available field data on the propagation and decay of semi-diurnal tidal water table waves in coastal aquifers. The symbols denote data obtained from the following sources: *Nielsen* [1990] (□); *Kang et al.*, [1994] (×); *Raubenheimer et al.*, [1999] (Δ); *Vigneaux* [2003] (◇) and *Cartwright* [2004] (+).

2. Existing Single Length Scale Solutions

Nielsen [1990] and *Li et al.* [2000a] derived perturbation solutions to the linearised 1D Boussinesq equation using the perturbation parameter,

$$\varepsilon = kA \cot \beta \quad (1)$$

where A is the (simple harmonic) forcing amplitude, β is the boundary slope and k is the (single valued) water table Boussinesq wave number,

$$k = k_B = \sqrt{\frac{n_e \omega}{2KD}} \quad (2)$$

where n_e is the effective porosity, K is the hydraulic conductivity, ω is the oscillation frequency and D is the mean aquifer thickness. That is, both the amplitude decay rate and the rate of increase in phase lag are accounted for by a single length scale. The difference between the two solutions is that *Nielsen* [1990] matched the sloping boundary condition

approximately whilst satisfying the flow equation in the interior exactly. *Li et al.* [2000a] however, transformed the governing equation to a fixed boundary problem enabling an exact match of the boundary condition but only approximately satisfying the flow equation in the interior.

More recently *Teo et al.* [2003] and *Jeng et al.* [2005a,b,c] have extended perturbation solutions of the Laplace equation and kinematic free surface boundary condition to higher orders. These new solutions have broadened the range of validity of the perturbation approach by adopting a range of different perturbation parameters. However, in the derivation of all of these solutions the spatial coordinate x is non-dimensionalised with the ‘linear decay length’ $(1/k_B)$,

$$L = \sqrt{\frac{2KD}{n_e \omega}} \quad (3)$$

thereby also imposing the same, single valued length scale for amplitude decay and phase shift which is in contradiction with the data (cf. Figure 2).

3. A New Dual Length Scale Solution

In a step towards obtaining a solution which at least qualitatively accounts for the dual length scales observed in the data (cf. Figure 2), a new solution of the 1D Boussinesq equation is derived here based on the same successive approximations approach as *Nielsen* [1990]. That is, the boundary condition is matched approximately whilst satisfying the interior flow equation exactly. Allowing for $k_r \neq k_i$ (where $k = k_r + ik_i$) yields the following dual length scale solution,

$$\begin{aligned} h(x,t) = & D + Ae^{-k_{1\omega,r}x} \cos(\omega t - k_{1\omega,i}x) \\ & + A^2 \cot \beta \left[\frac{k_{1\omega,r}}{2} + \frac{|k_{1\omega}|}{2} e^{-k_{2\omega,r}x} \cos(2\omega t + \text{Arg}\{k_{1\omega}\} - k_{2\omega,i}x) \right] \\ & + \frac{A^3 \cot^2 \beta}{4} \left[\begin{aligned} & \left\{ -k_{1\omega,i}k_{2\omega,r} - k_{1\omega,r}k_{2\omega,i} + k_{1\omega,r}k_{1\omega,i} \right\} e^{-k_{1\omega,r}x} \sin(\omega t - k_{1\omega,i}x) + \\ & \left\{ \frac{3}{2}k_{1\omega,i}^2 - \frac{3}{2}k_{1\omega,r}^2 + k_{1\omega,r}k_{2\omega,r} + k_{1\omega,i}k_{2\omega,i} \right\} e^{-k_{1\omega,r}x} \cos(\omega t - k_{1\omega,i}x) + \\ & \left\{ -k_{1\omega,i}k_{2\omega,r} - k_{1\omega,r}k_{2\omega,i} + k_{1\omega,r}k_{1\omega,i} \right\} e^{-k_{3\omega,r}x} \sin(3\omega t - k_{3\omega,i}x) + \\ & \left\{ \frac{1}{2}k_{1\omega,i}^2 - \frac{1}{2}k_{1\omega,r}^2 + k_{1\omega,r}k_{2\omega,r} - k_{1\omega,i}k_{2\omega,i} \right\} e^{-k_{3\omega,r}x} \cos(3\omega t - k_{3\omega,i}x) \end{aligned} \right] + H.O.T \end{aligned} \quad (4)$$

where the wave number $k_{m\omega} = k_{m\omega,r} + ik_{m\omega,i}$ and the subscripts $m\omega$ ($m = 1..3$) denote the m -th harmonic component; r and i denote the real and imaginary parts, respectively and $H.O.T$ are higher order terms.

4. Model Application

In this following analysis, the existing solutions that have been tested along with equation (4) are: equation 32 of *Nielsen* [1990]; equation 15 of *Li et al.* [2000a] and equation 34 of *Teo et al.* [2003].

In the analysis of their data, *Cartwright et al.* [2004] describe in detail the inability of current small-amplitude water table wave dispersion theory [e.g. *Barry et al.*, 1996; *Nielsen et al.*, 1997; *Li et al.*, 2000b] to predict the observed water table wave dispersion despite considering both finite-depth (vertical flow) and capillarity effects. As a consequence, all of the solutions described above are applied here in a ‘quasi-predictive’ manner. That is, the required input wave numbers used are those obtained from the observed decay and phase shifts in the interior. The aim of the present exercise is to illustrate the importance of allowing for the two length scales as seen in the data. The wave numbers estimated from the sand flume experiment of *Cartwright et al.* [2004] are given in Table 1.

Table 1. Experimental wave numbers from *Cartwright et al.* [2004]. R is the regression coefficient.

	$z = 0.8\text{m}$	R^2
$k_{1\omega, r} =$	0.584	0.998
$k_{1\omega, i} =$	0.343	0.995
$k_{2\omega, r} =$	0.779	0.998
$k_{2\omega, i} =$	0.311	0.974
$k_{3\omega, r} =$	0.781	0.988
$k_{3\omega, i} =$	0.208	0.916
$k_{4\omega, r} =$	0.713	0.869
$k_{4\omega, i} =$	0.378	0.837

Aside from equation (4), all of the existing solutions rely on a single input length scale ($L = 1/k$) to account for both the amplitude decay rate and rate of increase in phase lag (cf. section 0). As the available data indicate that $k_r \neq k_i$ (see Figure 2) we set the input wave number for these three models as the average of the real and imaginary parts of the first harmonic. That is, $k = 0.464$ for $k_{1\omega} = 0.584 + 0.343i$.

For all of the solutions, the solution is only valid landward of the shoreline location ($x > x_{SL}$) at any given time. For locations that are at or seaward of the shoreline ($x \leq x_{SL}$) at any given time the head is assumed to be hydrostatic and equal to the driving head, i.e.,

$$\begin{aligned} \eta(x, t) &= \eta_{\text{solution}}(x, t) && \text{for} && x > x_{SL}(t) \\ \eta(x, t) &= A \cos \omega t && \text{for} && x \leq x_{SL}(t) \end{aligned} \quad (5)$$

The origin of the fixed coordinate system was set as the mid point of the forcing zone i.e. $x_0 = 1.44$ m. The driving head parameters are provided in Table 2.

Table 2. Summary of the driving head parameters, D is the mean level, A is the amplitude and T is the period of oscillation.

D	A	T
1.01	0.204	348
[m]	[m]	[sec]

5. Comparison with Experimental Observations

To facilitate comparison of each of the solution's ability to replicate the observed amplitude decay rate and rate of increase in phase lag, harmonic analysis was used to extract the harmonic amplitude and phases from the analytical results for comparison with the experimental data.

5.1. Amplitudes

Figure 3 shows that all the solutions under-predict the amplitude decay rate of the first harmonic (\circ) in the forcing zone as a direct result of their neglect of seepage face formation. In all the models the exit point of the groundwater water table is assumed to be always coupled with the shoreline and as a consequence larger amplitudes are seen relative to the results of experiments where the water table exit point was decoupled from the shoreline during the low 'tide' part of the forcing [cf. *Cartwright et al.*, 2004; 2005].

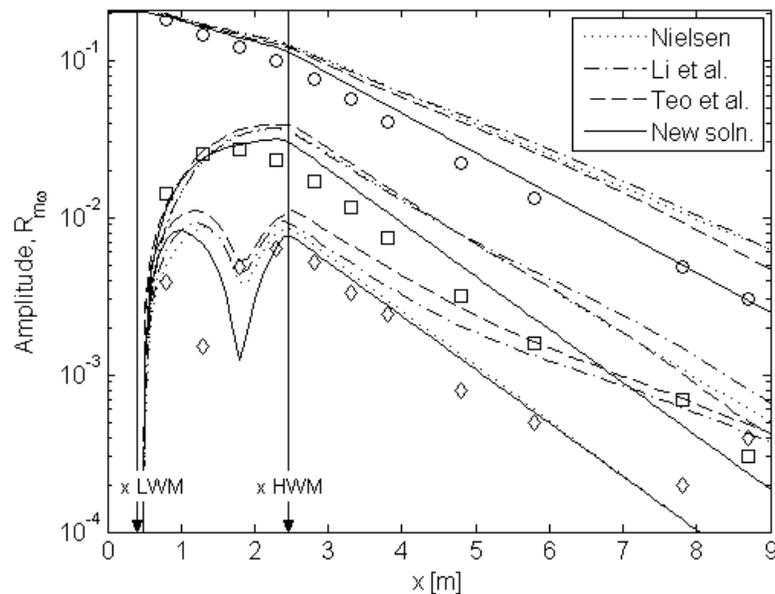


Figure 3. Amplitude profiles extracted from the data for the first (\circ) and third (\diamond) harmonics, compared with those generated by the various analytical solutions indicated in the legend.

In the interior, the shortcomings of the three single length scale solutions are clearly evident; they all fail to accurately predict the amplitude decay profile. In contrast, the new dual length scale solution matches the data reasonably well.

Also in the interior, the influence of the higher order terms in the *Li et al.* [2000a] and *Teo et al.* [2003] solutions are seen as deviations from a purely exponential decay, i.e. deviations from a straight line in the log-linear plot. This influence becomes more apparent in the higher harmonic profiles, in particular for the third harmonic (ψ), where substantial curvature is observed. This discrepancy is probably due to the fact that although the solutions match the boundary condition exactly, they only approximately match the interior flow equation.

Upon the application of equation (5), each of the solutions illustrates the generation of higher harmonics in the forcing zone that is in reasonable agreement with the data. However small differences are present which reveal some insight into the processes occurring at the boundary. In the case of the second harmonic (ϕ), each of the solutions predicts a maximum amplitude at the high water mark whereas the maximum in the data occurs near the mid point of the forcing zone. The difference here was shown by *Cartwright et al.* [2005] to be due to the presence of a seepage face in the experiment that is neglected in all of the analytical solutions. It may well be argued that the differences seen between the solutions and the data in this regard is only small, however, in the field where the extent of seepage faces may be substantially greater, neglect of their presence is likely to affect the analytical solutions' ability to accurately predict the generation of higher harmonics.

In the case of the third harmonic (ψ), the new dual length scale solution provides much better agreement with the observed amplitude profile as it uses the observed dispersive properties of the higher harmonic components. *Nielsen's* [1990] also performs well but the other two solutions, which satisfy the governing flow equation only approximately in the interior, show significant deviations from an exponential decay. All solutions do reasonably well at qualitatively reproducing the observed initial decrease in amplitude to a minimum near the mid point of the forcing zone before rising to a maximum at the high water mark. Similar differences to those observed in relation to the second harmonic (ϕ) are seen in that the solutions depict a minimum closer to the high water mark than is observed in the data, again due to the neglect of the seepage face [cf. *Cartwright et al.*, 2005].

5.2. Phases

The comparison of theoretical and observed phase profiles shown in Figure 4 also highlights the limitations of the single length scale solutions. All three solutions under predict the phase speed in the interior whereas the dual length scale solution does well at reproducing the development of the phase lag in the case of the first and second harmonics. Evidence of the higher order terms contained in the *Li et al.* [2000a] and *Teo et al.* [2003] solutions is again seen as deviations from the expected straight line. Each of the solutions reproduces the generation of the second harmonic in the forcing zone reasonably well. All four solutions perform poorly in predicting the phase behaviour of the third harmonic; however, the dual length scale model is able to reproduce the rate of increase in phase lag in the interior.

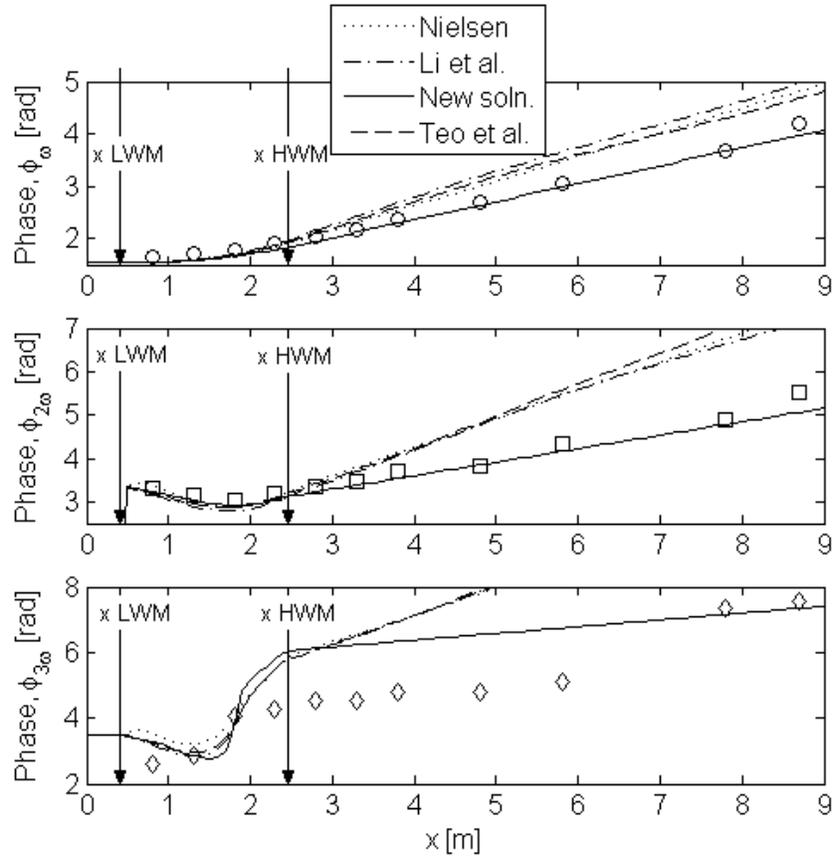


Figure 4. Phase lag profiles extracted from the data for the first (a), second (b) and third (c) harmonics compared with those generated by the analytical solutions.

6. Conclusion

A compilation of field data from the literature (see Figure 2) clearly shows the presence of two length scales in relation to the propagation of water table waves. That is, the length scale associated with the amplitude decay rate (k_r) is not equal to that for the rate of increase in phase lag (k_i) as is commonly assumed in the derivation of analytical solutions to the sloping boundary problem [e.g. *Nielsen*, 1990; *Li et al.*, 2000a; *Teo et al.*, 2003; *Jeng et al.*, 2005a,b,c].

Three existing single length scale solutions to the sloping boundary problem have been tested against observations from a simple laboratory aquifer. As a result of the assumption of a single length scale, these three solutions are unable to accurately reproduce the observed differences between both the amplitude decay rate and the rate of increase in phase lag in the interior.

In a step towards overcoming this limitation, a new dual length scale solution has been derived. When tested against the data, the new solution has been shown to accurately reproduce both the amplitude decay rate and the rate of increase in the phase lag for all of the

first three harmonics. Since the input wave numbers have been derived from the data, such a result is expected however, the fact that all of the single length scale solutions are unable to reproduce the observations, despite using an average of the experimental wave number, clearly illustrates the importance of incorporating the two different length scales into the solution of the problem.

Based on this finding we recommend that further mathematical development of the sloping boundary problem based on the use of a single length scale (i.e. assuming that $k_r = k_i$) should be avoided and that some consideration of the duality in length scales be incorporated.

The reader is reminded that each of the solutions above have been applied in a quasi-predictive manner only, using experimental wave numbers as input as opposed to theoretically predicted values. This has been due to the large discrepancies between the observed dispersion of the pressure wave and that predicted by currently available dispersion relation theory as analysed in detail by *Cartwright et al.* [2004]. Future research should be carried out to improve the understanding of the physical processes that determine $k = k_r + ik_i$, where $k_r \neq k_i$.

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