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Measuring measurement–disturbance relationships with weak values

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Abstract. Using formal definitions for the measurement precision ϵ and the disturbance (measurement back-action) η , Ozawa (2003 *Phys. Rev. A* **67** 042105) has shown that Heisenberg's claimed relation between these quantities is false in general. Here, we show that the quantities introduced by Ozawa can be determined experimentally, using no prior knowledge of the measurement under investigation—both quantities correspond to the root-mean-squared difference given by a weak-valued probability distribution. We propose a simple three-qubit experiment that can illustrate the failure of Heisenberg's measurement–disturbance relation and the validity of an alternative relation proposed by Ozawa.

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1. Introduction

The Heisenberg–Robertson uncertainty relation [1, 2] constrains the standard deviations (SDs) of two arbitrary non-commuting observables A and B ²:

$$\sigma(A)\sigma(B) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|. \quad (1)$$

The foremost example is for canonically conjugate observables, $\sigma(q)\sigma(p) \geq \hbar/2$, as considered by Heisenberg [1]. The uncertainty relation of equation (1) is now uncontroversial and is well verified experimentally [3, 4]. However, this was not the only relation introduced by Heisenberg in [1]; the first relation discussed there involved quite different quantities: ϵ , the *precision* with which a quantity is measured, and η , the amount of *disturbance* (Heisenberg called this the *discontinuous change*) in some other quantity. The status of this measurement–disturbance relation (MDR) has been a matter of considerable debate [5]–[7].

Heisenberg at first considered only position measurement and momentum disturbance, and postulated the relation $\epsilon(q)\eta(p) \gtrsim \hbar$, giving a heuristic derivation from a description of Compton scattering [1]. Later [8], he derived the MDR

$$\epsilon(q)\eta(p) \geq \hbar/2, \quad (2)$$

but only for a very special case [9]–[11], viz where initially the particle was in a momentum eigenstate (and thus had completely undefined position) and the measurement apparatus performs a minimally disturbing measurement [12]. In this case, the measurement precision $\epsilon(q)$ can be identified with the post-measurement position SD $\sigma'(q)$, and the momentum disturbance can be quantified by the post-measurement momentum SD $\sigma'(p)$, so that equation (2) follows from equation (1). For other types of initial states or measurement apparatus, it is not at all obvious how these quantities should be defined [7, 9], or even that the analogue of equation (2) should be expected to hold [13].

It was argued by Scully *et al* [5] that, in the context of a twin-slit interferometer, one can perform a position measurement with sufficient precision to determine which way the particle goes, without disturbing its momentum at all. In such experiments, $\eta(p)$ is indeed zero, if one defines this quantity as the root-mean-squared (rms) difference of the so-called weak-valued probability distribution for momentum disturbance [14]. This is a distribution that can be, and indeed recently has been [15], directly observed experimentally. It thus seems that the MDR of equation (2) is not valid in general.

This conclusion, that an MDR of the Heisenberg form was not universally valid, was independently arrived at by Ozawa [16]. But Ozawa went further, and proposed a new, universally valid, MDR [16]. Moreover, Ozawa’s MDR applies for any pair of observables A (which is measured) and B (which is disturbed). However, unlike in [14], the quantities of Ozawa’s MDR were defined purely theoretically. That is, no prescription was given in [16] on how these quantities could be experimentally determined given a black-box apparatus that performs some sort of measurement of A , on a system prepared in some fixed but *unknown mixed state* ρ . In a later paper [10], Ozawa gave a method for determining the measurement precision ϵ ; however, this method requires the system to be prepared in a *known pure state* $|\psi\rangle$.

² These SDs would thus be determined by distinct measurements on two distinct sub-ensembles of an ensemble of identically prepared systems.

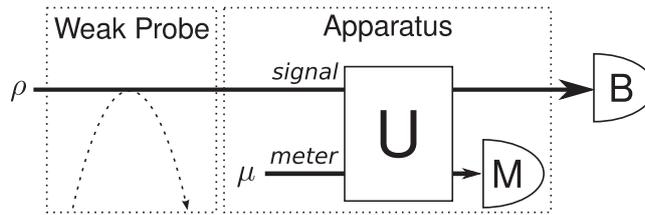


Figure 1. Schematic diagram of the procedure using weak measurements to extract the measurement disturbance and precision quantities. An initial weak measurement shown in the left-hand box is used to gather information about the signal prior to its measurement by the apparatus. This apparatus can be described without loss of generality as the preparation of a meter state μ , the interaction of the signal and meter by a unitary U and the read-out of some meter observable M . Measurement results from the weak probe, and from the read-out of the meter or a final (third) measurement of the signal, are used to construct the measurement disturbance and precision quantities.

In this paper, we unify the approach of [14] with that of [16]. Firstly, we show that Ozawa's disturbance quantifier $\eta(B)$ is exactly the rms difference given by a weak-valued probability distribution for disturbance defined in [14]. Secondly, we show that Ozawa's precision quantifier $\epsilon(A)$ equals the rms difference given by another, analogously defined, weak-valued probability distribution. The remaining quantities in Ozawa's relation are simply the SDs of observables for the initial state ρ . Thus, Ozawa's MDR, for B -disturbance caused by an A -measurement, could be experimentally tested by an experimenter without knowledge of the initial system state, the initial meter state or the interaction. Finally, we propose a simple three-qubit experiment that could demonstrate 'interesting' cases where Ozawa's MDR would be validated but a Heisenberg-form MDR would be violated.

2. Ozawa's measurement–disturbance relation (MDR)

Using the measurement model described in figure 1, Ozawa's precision and disturbance quantities are defined as [16]

$$\epsilon(A) = \langle (U^\dagger(I \otimes M)U - A \otimes I)^2 \rangle^{1/2}, \quad (3)$$

$$\eta(B) = \langle (U^\dagger(B \otimes I)U - B \otimes I)^2 \rangle^{1/2}, \quad (4)$$

where the average is taken over the composite input signal and meter state. Here, U is the unitary evolution operator describing the complete interaction between the signal and meter, which is intended to imprint information about the signal observable A onto the meter observable M (which is read-out), whereas B is a signal observable that may (or may not) be disturbed as a result of the measurement of A . The quantity $\epsilon(A)$ is the rms difference between the initial value of A and the final value of M , which is the obvious definition for *measurement precision* [16]. Similarly, $\eta(B)$ is the rms difference between the initial and final values of B —the *disturbance* of B caused by measurement back-action.

One might expect that these quantities would satisfy a Heisenberg-form MDR

$$H \equiv \epsilon(A)\eta(B) \geq C(A, B), \quad (5)$$

where $C(A, B)$ derives from the commutator in equation (1),

$$C(A, B) \equiv \frac{1}{2} |\text{Tr} \{\rho[A, B]\}|. \quad (6)$$

Heisenberg's original relation equation (2) would then be a special case. However, as shown by Ozawa, this relation is false in general. Ozawa then suggested replacing the simple Heisenberg product H in equation (5) by a three-term expression involving not only $\epsilon(A)$ and $\eta(B)$, but also the SDs appearing in equation (1):

$$O \equiv \epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B). \quad (7)$$

This enabled him to derive a universally valid MDR [16]:

$$O \geq C(A, B) \quad (8)$$

Note that $\sigma(A)$ and $\sigma(B)$ are evaluated from measurements upon the initial signal state ρ .

3. Ozawa's MDR and weak values

Ozawa's precision and disturbance quantities both involve observables at two different times—before and after the signal–probe interaction. Classically, one could determine such a quantity simply by measuring the relevant observables at the two times. Quantum mechanically, the problem is of course that the first measurement would disturb the system, thus rendering the result of the second measurement irrelevant. This problem can be overcome by using weak measurements [17]. Such measurements may give vanishingly small disturbances of the underlying system, at the cost of having a very noisy result and so requiring a very large ensemble to obtain an accurate average. The average of a weak measurement result is particularly interesting when performed on an ensemble post-selected on a later, usually strong, measurement. Such averages are called weak values [17], and have been used to analyze a great variety of quantum phenomena [15], [18]–[26]. We will now show how these weak measurements can be utilized to extract the quantities in the universally valid MDR.

First consider the $\eta(B)$ quantity. For simplicity, we will take the eigenvalue spectrum $\{b\}$ of B to be discrete; the generalization to a continuous spectrum is not difficult to construct theoretically, but in an experiment discretization would be necessary [15]. We denote the projector onto the eigenspace associated with eigenvalue b as $\Pi(b)$. These projectors are Hermitian and hence are valid observables. The expectation value $\text{Tr}[\Pi(b)\rho]$ is the probability that the system would be found to have $B = b$, via a strong measurement of the observable $\Pi(b)$. The same number, between zero and one, would be obtained by averaging a weak measurement of $\Pi(b)$.

The equivalence between averages of weak and strong measurements of $\Pi(b)$ ceases when one considers post-selected ensembles. The general expression for a post-selected weak value, allowing for a mixed initial state, arbitrary evolution and an arbitrary final measurement, was given in [20]. In this case, the evolution in question is the interaction between the signal and the meter and, since we are interested in the disturbance to B , the final measurement we

wish to make is that of the observable B . The weak value of the projection observable $\Pi(b)$, post-selected on achieving a final result b_f , is

$${}_{b_f} \langle \Pi(b_i)_{\text{weak}} \rangle_\rho = \text{Re} \frac{\text{Tr} [\Pi(b_f) U \Pi(b_i) (\rho \otimes \mu) U^\dagger]}{\text{Tr} [\Pi(b_f) U (\rho \otimes \mu) U^\dagger]}. \quad (9)$$

Here, μ is the initial meter state, and $\Pi(b)$ is to be understood as $\Pi(b) \otimes I$. We have used subscripts i and f to explicitly indicate the values representing quantities before (initial) and after (final) the unitary evolution. The expression in equation (9) can be interpreted as the weak-valued probability of B initially taking the eigenvalue b_i , conditional on it finally taking the eigenvalue b_f . Hence, we will write this expression as $P_{\text{wv}}(b_i|b_f)$. Using this, it is possible to define a weak-valued *joint* probability distribution [14]³

$$\begin{aligned} P_{\text{wv}}(b_i, b_f) &= P_{\text{wv}}(b_i|b_f) P(b_f) \\ &= \text{Re} \left\{ \text{Tr} [\Pi(b_f) U \Pi(b_i) (\rho \otimes \mu) U^\dagger] \right\} \\ &= \text{Re} \langle U^\dagger \Pi(b_f) U \Pi(b_i) \rangle. \end{aligned} \quad (10)$$

Finally, we define a weak-valued probability distribution for a change δb in the value of observable B as

$$P_{\text{wv}}(\delta b) = \sum_{b_i} P_{\text{wv}}(b_i, b_f = b_i + \delta b) \quad (11)$$

$$= \sum_b \text{Re} \langle U^\dagger \Pi(b + \delta b) U \Pi(b) \rangle. \quad (12)$$

We will now show that the rms difference given by this weak-valued probability distribution for the change in B is identical to Ozawa's disturbance quantity $\eta(B)$. Making the signal and meter Hilbert spaces explicit, the mean-squared change in B is

$$\sum_{\delta b} (\delta b)^2 P_{\text{wv}}(\delta b) = \sum_{b, b'} (b' - b)^2 \text{Re} \langle U^\dagger [\Pi(b') \otimes I] U [\Pi(b) \otimes I] \rangle \quad (13)$$

$$= \langle (U^\dagger [B \otimes I] U - [B \otimes I])^2 \rangle, \quad (14)$$

where we have used the Hermitian operator identity $f(B) = \sum_b f(b) \Pi(b)$ and written out the real part using the sum of complex conjugate pairs in the last step. The final equation (14) is just the square of equation (4).

The measurement precision quantity $\epsilon(A)$ can be measured similarly. For each eigenvalue a , one performs a weak measurement of $\Pi(a)$ on the initial signal before it enters the apparatus, and then a strong measurement (read-out) of the final meter observable M , which contains information about A . As above, one can then construct the weak-valued probability distribution for the difference between the initial value of A and the final value of M . Since the meter is meant to measure A , we can assume, following [16], that the spectrum of M coincides with that of A . Then this weak-valued probability distribution can be evaluated to be the following (note the difference in the ordering of tensor products in the two terms):

$$P_{\text{wv}}(\delta a) = \sum_a \text{Re} \langle U^\dagger [I \otimes \Pi(a + \delta a)] U [\Pi(a) \otimes I] \rangle. \quad (15)$$

³ We note that the related formulae have been studied in [27]–[29].

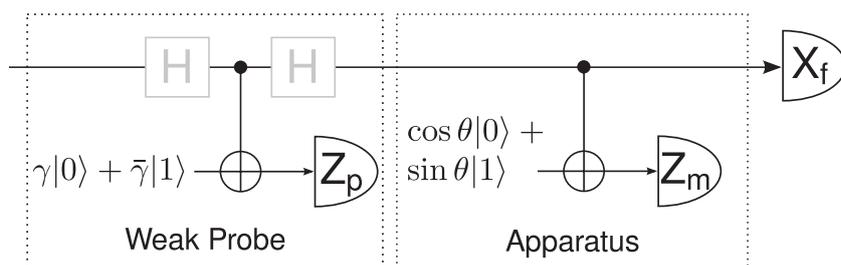


Figure 2. A qubit-based implementation of the device shown in figure 1. When measuring the disturbance, the probe measurement is conjugated by Hadamard operations (shown in gray) to perform a weak X -measurement. For precision, they are absent and the probe performs a weak Z -measurement.

Following a similar computation to the disturbance case,

$$\sum_{\delta a} (\delta a)^2 P_{\text{wv}}(\delta a) = \langle (U^\dagger [I \otimes M] U - [A \otimes I])^2 \rangle \quad (16)$$

is square of equation (3). Thus, both of Ozawa's quantities can be measured using a weak measurement immediately before the measurement under examination is applied, without any assumptions about ρ , μ or U . See figure 1.

A notable property of our experimental method of determining the measurement precision and disturbance quantities, via weak-valued probability distributions, is that they coincide with how a classical physicist might perform such a determination [30]. Of course classically one could just measure the initial value of the observable and the final value and then, after gathering statistics, find the rms difference. However, this method does not directly transfer to quantum mechanics because a strong projective measurement will disturb the statistics gathered. Using weak measurements gives a valid quantum mechanical procedure, which is also applicable to classical systems, in which case it would give exactly the same answer as would be obtained using strong measurements. This is another distinction from the experimental method proposed by Ozawa [10].

4. Qubit example

We will now construct an example using qubits. As usual [31], we write the Pauli matrices as X , Y and Z , and the states $|0\rangle$ and $|1\rangle$ denote Z eigenstates with eigenvalues of 1 and -1 , respectively.

We take the observable to be measured (the operator A) to be Z . The measurement apparatus is constructed by choosing the measurement interaction (the unitary U) to be the controlled-NOT (CNOT) operation [31], a pure initial meter state $\mu = |\theta\rangle\langle\theta|$, where $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, and the meter measurement observable (the operator M) to be Z ; see figure 2. The strength of this measurement can be quantified as $\cos 2\theta$, varying from a full strength Z -measurement at $\theta = 0$ and no measurement at $\theta = \pi/4$ [24]. To test the two MDRs (5) and (8), we consider the disturbance in the signal observable X (i.e. this is the operator B). This choice allows for the maximum value of the $C(A, B)$ of equation (6), which is the lower bound appearing in both MDRs. This maximum is achieved when the input signal state ρ is a Y eigenstate, which thus gives the most stringent tests for these MDRs.

The precision and disturbance quantities for this example are calculated from equations (3) and (4). They are

$$\begin{aligned}\epsilon^2(Z) &= \langle (U^\dagger(I \otimes Z)U - Z \otimes I)^2 \rangle = 4 \langle I \otimes |1\rangle\langle 1| \rangle, \\ \eta^2(X) &= \langle (U^\dagger(X \otimes I)U - X \otimes I)^2 \rangle = 2 \langle I \otimes (I - X) \rangle.\end{aligned}$$

For this particular measurement, both the precision and disturbance quantities are independent of the input state:

$$\epsilon(Z) = 2|\sin \theta|, \quad \eta(X) = \sqrt{2}|\cos \theta - \sin \theta|. \quad (17)$$

The product in the Heisenberg-form MDR (5) is thus

$$H = \epsilon(Z)\eta(X) = 2\sqrt{2}|\sin \theta||\cos \theta - \sin \theta|. \quad (18)$$

Now the pre-measurement uncertainties for an input Y eigenstate are

$$\sigma(X) = 1, \quad \sigma(Z) = 1. \quad (19)$$

Thus, the expression appearing in Ozawa's MDR is

$$\begin{aligned}O &= H + \epsilon(Z)\sigma(X) + \sigma(Z)\eta(X) \\ &= 2\sqrt{2}|\sin \theta||\cos \theta - \sin \theta| + 2|\sin \theta| + \sqrt{2}|\cos \theta - \sin \theta|.\end{aligned}$$

For both MDRs (5) and (8), the lower bound is

$$C(X, Z) = |\langle \Psi|[X, Z]|\Psi \rangle|/2 = |\langle \Psi|Y|\Psi \rangle| = 1. \quad (20)$$

It is easy to verify that the Heisenberg-form MDR $H \geq C(X, Z)$ is violated for all measurement strengths $\cos 2\theta$. On the other hand, Ozawa's universally valid MDR $O \geq C(X, Z)$ holds for all θ values, as expected. See figure 3.

To measure Ozawa's $\epsilon(A)$ and $\eta(B)$ quantities, it is necessary in general to use a series of different weak measurements on the initial signal, one for each eigenvalue of A or B (see equations (15) and (12), respectively), as done for $\eta(p)$ in [15]. However, the sum of the weak-valued probabilities will equal unity, and in the qubit case X and Z have only two eigenvalues. Thus only a single weak measurement is needed, so we can consider the observables X or Z themselves and use, for example,

$$2 \langle \Pi(x = \pm 1)_{\text{weak}} \rangle = 1 \pm \langle X_{\text{weak}} \rangle. \quad (21)$$

Thus the initial weak measurement can be performed using a measurement apparatus identical to that already defined above, based on a CNOT gate, as shown in figure 2. To avoid confusion with the *meter* state, we write the input *probe* state as $\gamma|0\rangle + \bar{\gamma}|1\rangle$. Here, $\gamma, \bar{\gamma} \in \mathbb{R}_+$, so that the measurement strength (which should be small) is $2\gamma^2 - 1$. The positive operator valued measure (POVM) elements [31] corresponding to the two outcomes $Z_p = \pm 1$ from reading out the probe are

$$E_{\pm} = \frac{1}{2} [1 \pm (2\gamma^2 - 1)O], \quad (22)$$

where $O = Z$ or X as appropriate (see figure 2).

Now we will describe how the data from this experimental arrangement are processed to yield $\epsilon(Z)$ and $\eta(X)$. First consider the disturbance quantity $\eta(X)$. There are only two non-zero terms in equation (13):

$$\eta^2(X) = 4P_{\text{wv}}(\delta X = +2) + 4P_{\text{wv}}(\delta X = -2). \quad (23)$$

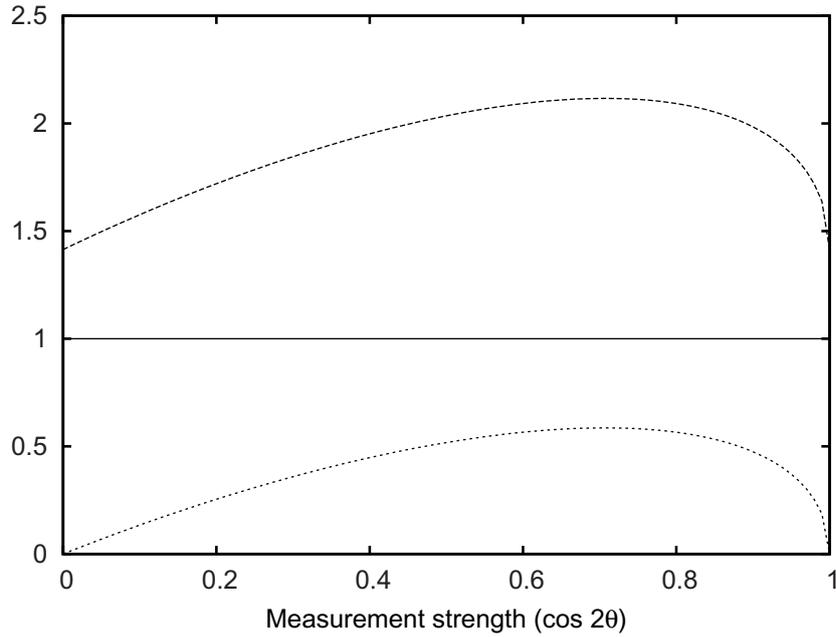


Figure 3. The quantities H (dotted line) and O (dashed line) appearing in the Heisenberg-form MDR and Ozawa's universally valid MDR for the example qubit model. Both have the same lower bound $C(X, Z)$, which equals 1 here (solid line). In this case, the Heisenberg-form MDR is false for all measurement strengths.

Consider the $P_{\text{wv}}(\delta X = +2)$ term. From equation (11),

$$P_{\text{wv}}(\delta X = +2) = P_{\text{wv}}(X_i = -1 | X_f = 1) P(X_f = 1).$$

The last factor equals the directly measurable probability from the final signal read-out, which equals $1/2$ for the system in question. The first factor is a *weak-valued* probability. From equation (21), it can be computed from directly measured joint probabilities as follows:

$$2P_{\text{wv}}(X_i = \pm 1 | X_f) = 1 \pm \frac{\sum_{Z_p} Z_p P(Z_p | X_f)}{2\gamma^2 - 1}. \quad (24)$$

As above, $Z_p \in \{-1, 1\}$ is the result of the read-out of the weak probe, which effects the measurement as described by the POVM in equation (22) with $O = X$ in this instance. The precision quantity $\epsilon(Z)$ can be obtained in exactly the same way, by changing the probe interaction so that $O = Z$ in equation (22) and by replacing the final read-out of the signal X_f by a read-out of the meter Z_m .

To evaluate the feasibility of performing the proposed experiment, we have performed a numerical simulation for imperfect CNOT gates, with results shown in figures 4 and 5. The results in figure 4 plot the H quantity showing a violation of the Heisenberg-form MDR, and the results in figure 5 plot the O quantity validating Ozawa's MDR. In this simulation, the CNOT gates are replaced by non-ideal CNOT operations. The non-ideal CNOT consists of the ideal CNOT gate with probability $1 - p$, the identity with probability $p/2$ and a swap gate [31] with probability $p/2$. This model is motivated by considering mode mismatch in a standard experimental implementation of a CNOT gate in linear optics [32]–[34].

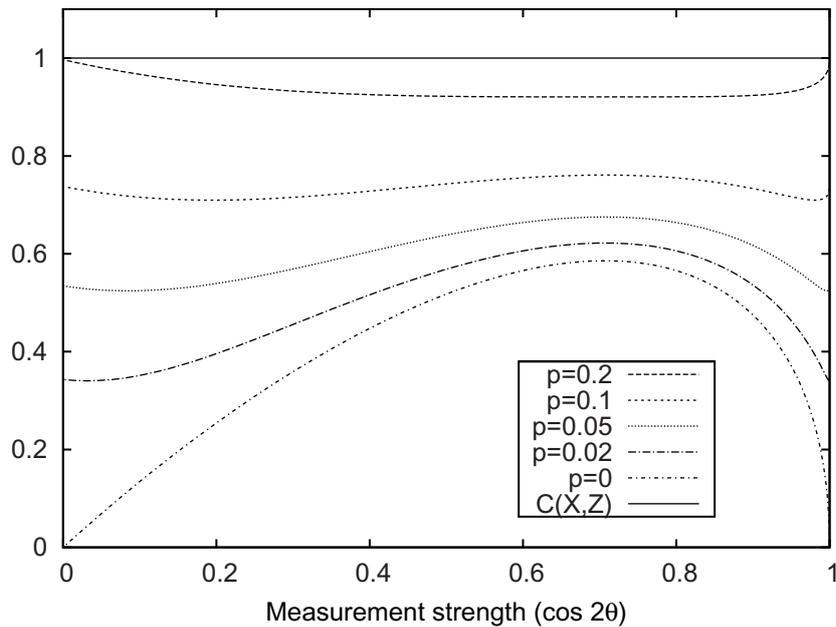


Figure 4. The H quantity from the Heisenberg-form MDR evaluated numerically for the proposed experiment given in figure 2 with each of the two CNOT gates replaced by non-ideal CNOT operations parameterized by an error rate p (see the text). A range of values of p between 0 and 0.2 are shown (see the legend) with the values of H tending to increase as p increases. The lower bound $C(X, Z)$ is the same as in figure 3 (solid line). All of the curves shown here violate the lower bound of the Heisenberg-form MDR.

These simulations show that for the error model considered, the quantities of interest are independent of the probe measurement strength. They also show that to demonstrate results similar to the ideal (in particular, clearly showing a violation of the Heisenberg-form MDR) requires error rates of 10% or less for both CNOT gates. This should be feasible in a number of platforms for quantum information processing.

5. Conclusion

In this paper, we have proposed a method for experimentally determining Ozawa's measurement precision $\epsilon(A)$ and disturbance $\eta(B)$ quantities, using weak measurements. This method does not require prior information about how the initial system state is prepared, or about how the measurement apparatus under study operates (except for how its outcomes encode a measurement of the relevant system observable A). Moreover, this method is understandable classically and works for classical as well as quantum systems.

Using this approach, we suggest an example experiment using qubits which would validate Ozawa's MDR while showing that the Heisenberg-form MDR is violated. We have also performed numerical simulations to quantify the operation of this three-qubit scheme under an error model motivated by mode mismatch in a linear optical quantum computing realization.

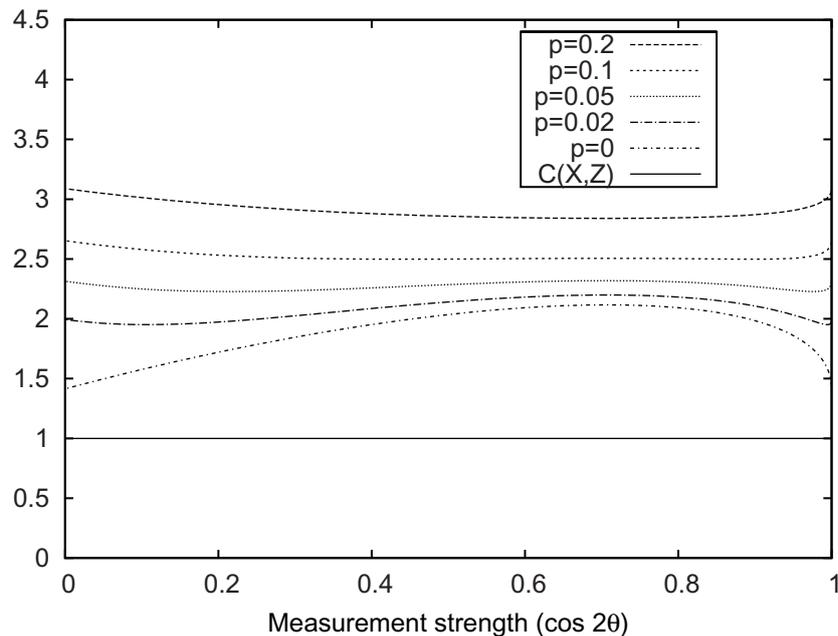


Figure 5. The O quantity from Ozawa's MDR plotted in the same manner as figure 4. The values of O tend to increase as p increases (see the legend). All of the curves shown here obey the lower bound of Ozawa's MDR.

Such an experiment, quantifying the MDR, would address a much-discussed issue dating back to the birth of modern quantum mechanics.

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