

Information erasure without an energy cost

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Landauer argued that the process of erasing the information stored in a memory device incurs an energy cost in the form of a minimum amount of mechanical work. We find, however, that this energy cost can be reduced to zero by paying a cost in angular momentum or any other conserved quantity. Erasing the memory of Maxwell's demon in this way implies that work can be extracted from a single thermal reservoir at a cost of angular momentum and an increase in total entropy. The implications of this for the second law of thermodynamics are assessed.

Keywords: information erasure, thermodynamics, canonical ensemble, spin system

1. Introduction

The idea of a link between information and thermodynamics can be traced back to Maxwell's famous Demon, a supposed microscopic intelligent being, the actions of which might present a challenge to the second law of thermodynamics (Maxwell 1871, Leff & Rex 1990, 2003). This idea was made quantitative by Szilard (1929) who showed, by means of a simple one-molecule gas, that information acquisition, for example by a Maxwell Demon, is necessarily accompanied by an entropy increase of not less than $k \ln(2)$, where k is Boltzmann's constant. A closely related phenomenon is the demonstration, due to Landauer (1961), that erasing an unknown bit of information requires heat to be dissipated, amounting to not less than $kT \ln(2)$, where T is the temperature of a thermal reservoir (Plenio & Vitelli 2001, Maruyama *et al.* 2010).

Changing the temperature of the reservoir changes the energy cost of erasure and so it might be argued that it is the fixed entropy of $k \ln(2)$ that is the fundamental cost of erasing one bit. One might then be tempted to argue, erroneously, that energy is not a consideration at all. However the use of thermodynamic reservoirs at some temperature is explicit in Szilard's and Landauer's work. Also, the term $k \ln(2)$ represents *thermodynamic entropy* where the dimension of k is energy per temperature. An energy cost is therefore inescapable in these analyses except in the extreme case of zero temperature.

We show here, however, that it is possible to avoid an energy cost, irrespective of the temperature (Vaccaro & Barnett 2009). All that is required is a different kind of reservoir such as one based on angular momentum rather than on energy. We begin by recasting Landauer's erasure in terms of quantum information theory. We then introduce a spin reservoir and show how it can be used to erase information at a cost in terms of angular momentum and without a cost in energy and conclude with a discussion.

2. Erasure using a thermal reservoir

In quantum information theory, the fundamental component is not the bit, but rather the qubit (Nielsen & Chuang 2000, Barnett 2009). A qubit may be any quantum system with two distinct states, which we label $|0\rangle$ and $|1\rangle$. The distinguishing feature of a qubit, of course, is that it can be prepared in any superposition of these two states. It is instructive to recast Landauer's erasure principle for a qubit using what we shall refer to as *Model A* of qubit erasure. Let us suppose that the qubit is in an initially unknown state and that we wish to reset it by forcing it into the state $|0\rangle$. Let the two states $|0\rangle$ and $|1\rangle$ be initially degenerate, with energy 0. We can erase the qubit state by placing it in contact with a reservoir at temperature T and then inducing an energy splitting between the qubit states so that $|0\rangle$ has energy 0, but the state $|1\rangle$ has energy E . The splitting is induced adiabatically, that is, sufficiently slowly that the qubit remains in thermal equilibrium with the thermal reservoir. The state of the qubit when the energy splitting is E is governed by the Boltzmann (or maximum entropy) distribution with density operator

$$\rho = \frac{|0\rangle\langle 0| + e^{-E/kT}|1\rangle\langle 1|}{1 + e^{-E/kT}}. \quad (2.1)$$

The work required to increase the splitting from E to $E + dE$ while in contact with the reservoir is given the probability of occupation of the state $|1\rangle$ multiplied by dE , that is $dW = e^{-E/kT}(1 + e^{-E/kT})^{-1}dE$. The total work in increasing the splitting from zero to infinity is $W = \int dW = kT \ln 2$ and the state of the qubit is $|0\rangle$ as expected. The qubit is removed from the reservoir and the energy degeneracy then restored. The erasure here is driven by maximising the entropy subject to conservation of energy as the energy gap of the states of the memory qubit grows.

3. Erasure using a spin reservoir

Consider a second model of qubit erasure, *Model B*, in which the qubit logic states $|0\rangle$ and $|1\rangle$ are associated with different eigenvalues of a conserved observable other than energy. For definiteness, we take this observable to be the z component of angular momentum and the qubit to be a spin- $\frac{1}{2}$ particle. Let the reservoir be constructed of similar particles. In Model A we were able to increase the energy splitting between the logical states $|0\rangle$ and $|1\rangle$ of the memory qubit by some external means, and thermalisation with the reservoir involved the exchange of energy. The erasure of the memory qubit in Model B, however, proceeds via the exchange of discrete quanta of angular momentum with the new reservoir. Conservation of angular momentum requires the memory qubit and reservoir to have compatible angular momentum splittings. We keep the reservoir splittings fixed and change the effective angular momentum splittings of the memory system. For this we use a supply of ancilla spin- $\frac{1}{2}$ particles which are initially in the logical zero state, and a specific controlled operation that ensures the combined memory and ancilla system is in a superposition of two states which are separated in angular momentum by a given multiple of \hbar . To emphasise their physical context, we shall hereafter refer to the qubits in Model B as spins.

(a) Construction of the spin reservoir

Let the reservoir consist of N spin- $\frac{1}{2}$ particles whose spatial degrees of freedom are in thermal equilibrium with a much larger heat bath at temperature T . In contrast, the internal spin degrees of freedom are in an *independent* equilibrium state. The decoupling between the spatial and spin degrees of freedom may be ensured by requiring the internal spin states of each spin to be degenerate in energy. The spatial degrees of freedom are described by a probability distribution which depends on the temperature of the heat bath. The temperature and spatial degrees of freedom requires no further consideration.

We describe the internal spin degree of freedom of each spin in terms of the eigenstates of the z component of spin angular momentum, S_z , which we label in the logical basis $|0\rangle$ and $|1\rangle$. Here $|i\rangle$ represents an eigenstate of S_z with eigenvalue $(i - \frac{1}{2})\hbar$. The internal states of the collection of spins in the reservoir is then given by $\bigotimes_{i=1}^N |x_i\rangle$ where $x_i = 0, 1$. Consider the set of these states for which $\sum_{i=1}^N x_i = n$ is an integer in the range $0 \leq n \leq N$. Each of these states has a J_z eigenvalue of $(n - N/2)\hbar$ and there are $\binom{N}{n}$ such states, where J_z is the z component of the combined spin angular momentum of the whole reservoir. We label elements of this set as the collective state $|n, \nu\rangle_r$ where $\nu = 1, 2, \dots, \binom{N}{n}$ uniquely labels each state in the set. Thus a convenient basis for the state space of the spin degrees of freedom of the reservoir is given by $\{|n, \nu\rangle_r : n = 0, \dots, N; \nu = 1, 2, \dots, \binom{N}{n}\}$.

We imagine that the internal spin state of the reservoir is generated and maintained by interacting through the exchange of spin angular momentum with a much larger “spin bath”. The spin bath consists of M spin- $\frac{1}{2}$ particles with a basis set $\{|m, \mu\rangle_b : m = 0, \dots, M; \mu = 1, 2, \dots, \binom{M}{m}\}$. Here the collective state of the bath spins $|m, \mu\rangle_b$ represents the μ -th eigenstate of J_z with eigenvalue $(m - M/2)\hbar$ where $\mu = 1, 2, \dots, \binom{M}{m}$ and is defined analogously to that of the reservoir state $|\cdot, \cdot\rangle_r$. The spin bath maintains the reservoir in a state such that the average z component of spin of the reservoir is given by $\langle J_z \rangle = (\alpha - \frac{1}{2})N\hbar$ for $0 \leq \alpha \leq 1$.

(b) Information Erasure

We treat the reservoir as a canonical ensemble, but instead of energy being exchanged between the reservoir and spin bath, as is the usual case, the systems randomly exchange z component of spin according to the mapping $|n, \nu\rangle_r |m, \mu\rangle_b \leftrightarrow |n \pm 1, \nu'\rangle_r |m \mp 1, \mu'\rangle_b$ where $\nu' = 1, 2, \dots, \binom{N}{n \pm 1}$ and $\mu' = 1, 2, \dots, \binom{M}{m \mp 1}$. At equilibrium, the probabilities $P_{n, \nu}$ of finding the reservoir in the state $|n, \nu\rangle_r$ is given by maximising the *information-theoretic entropy* $(-\sum_{n, \nu} P_{n, \nu} \ln P_{n, \nu})$ with respect to $P_{n, \nu}$ subject to the constraints $\sum_{n, \nu} n P_{n, \nu} = \alpha N$ and $\sum_{n, \nu} P_{n, \nu} = 1$. The equilibrium probability distribution is found to be

$$P_{n, \nu} = \frac{e^{-n\gamma\hbar}}{(1 + e^{-\gamma\hbar})^N} \quad (3.1)$$

where $\gamma = \ln[(1 - \alpha)/\alpha]/\hbar$.

Now let us suppose another spin- $\frac{1}{2}$ particle is our memory qubit. It begins in the maximally mixed state $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ and we wish to erase its memory and leave it in the logical zero state $|0\rangle\langle 0|$. Let there be a large collection of ancillary spin- $\frac{1}{2}$ particles in the $|0\rangle\langle 0|$ state for our use. The first stage entails putting the memory

spin in spin-exchange contact with the reservoir, letting the combined reservoir-memory spin system come to equilibrium, and then separating the memory spin from the reservoir. At this point the state of the memory spin is

$$p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| \quad (3.2)$$

with $p_1 = e^{-\gamma\hbar}/(1 + e^{-\gamma\hbar}) = 1 - p_0$. We have assumed that the value of $\langle J_z \rangle = (\alpha - \frac{1}{2})N\hbar$ is maintained by the spin bath and so the value of γ remains fixed despite the contact with the memory spin.

In model A the energy splitting of the memory qubit is slowly increased while the qubit is in equilibrium with a thermal reservoir. We want the same principle to operate for Model B but with angular momentum in place of energy. In Model B, the z component of angular momentum of the two states that represent the memory are separated in value by \hbar . This separation can be increased by performing a controlled-not (CNOT) operation (Nielsen & Chuang 2000, Barnett 2009) on the memory and an ancilla spin as follows. The memory spin acts as the control, and the ancilla spin, which is initially in the state $|0\rangle\langle 0|$, acts as the target. The relevant properties of the CNOT operator U are given by

$$U|0j\rangle = |0j\rangle, \quad U|1j\rangle = |1k\rangle, \quad (3.3)$$

where j is 0 or 1, $k = 1 - j$ and, for convenience, $|xy\rangle$ represents the tensor product of the state $|x\rangle$ of the memory spin and $|y\rangle$ of the ancilla spin. After the CNOT operation the combined memory-ancilla state is $p_0|00\rangle\langle 00| + p_1|11\rangle\langle 11|$. The average angular momentum cost of this operation is $\hbar p_1 = \hbar e^{-\gamma\hbar}/(1 + e^{-\gamma\hbar})$. The combined memory-ancilla system is placed in random spin exchange contact with the reservoir while the reservoir-bath is undergoing random spin exchange as before. The spin exchange between the reservoir and the memory-ancilla system is constructed to leave all states unchanged except for the following mapping

$$|2, 1\rangle_r |00\rangle \leftrightarrow |0, 1\rangle_r |11\rangle \quad (3.4)$$

where $|n, \nu\rangle_r |ij\rangle$ represents the reservoir collective state $|n, \nu\rangle_r$ and memory-ancilla system state $|ij\rangle$. The random spin exchange continues for a sufficient time for the reservoir and memory-ancilla system to equilibrate. The state is then given by

$$p_0|00\rangle\langle 00| + p_1|11\rangle\langle 11| \quad (3.5)$$

where now $p_1 = e^{-2\gamma\hbar}/(1 + e^{-2\gamma\hbar}) = 1 - p_0$ from Eq. (3.1). This completes the first cycle.

Another ancilla spin is added and a CNOT operation is performed as before to yield the state $p_0|000\rangle\langle 000| + p_1|111\rangle\langle 111|$ with a spin cost of $\hbar e^{-2\gamma\hbar}/(1 + e^{-2\gamma\hbar})$. The combined memory-ancilla system put in spin-exchange contact with the reservoir with the mapping

$$|3, 1\rangle_r |000\rangle \leftrightarrow |0, 1\rangle_r |111\rangle. \quad (3.6)$$

This process is repeated. After n cycles, the memory-ancilla spins are in the logical zero state and the logical 1 state with probabilities p_0 and p_1 where $p_0 = e^{-n\gamma\hbar}/(1 + e^{-n\gamma\hbar}) = 1 - p_0$ according to Eq. (3.1). In the limit of many repetitions and large

N , the memory-ancilla system approaches a pure state where each spin is in the logical zero state. The total spin cost of the whole process is

$$\Delta J_z = \sum_{n=1}^{\infty} \hbar \frac{e^{-n\gamma}}{1 + e^{-n\gamma}}. \quad (3.7)$$

This sum is bounded by

$$\gamma^{-1} \ln(1 + e^{-\gamma\hbar}) < \Delta J_z < \gamma^{-1} \ln(2). \quad (3.8)$$

If we include the spin of the initial state, then the cost is

$$\Delta J'_z = \sum_{n=0}^{\infty} \hbar \frac{e^{-n\gamma\hbar}}{1 + e^{-n\gamma\hbar}} = \Delta J_z + \frac{1}{2} \hbar, \quad (3.9)$$

$$\gamma^{-1} \ln(2) < \Delta J'_z < \gamma^{-1} \ln(1 + e^{\gamma\hbar}). \quad (3.10)$$

Clearly the costs associated with erasure depend on the physical qubit, and need not include an energy term, in contradistinction to the suggestion of Landauer and many others.

(c) *Absence of an energy cost*

There are two important points to be made about the absence of an energy cost in Model B. First, the cost of erasure is in terms of the quantity defining the logic states, which is spin angular momentum. This cost arises because the CNOT operation in Eq. (3.3) does not conserve this angular momentum. In contrast, the conservation of energy is trivially satisfied due to the energy degeneracy.

For the second point, note that the energy-free cost of Model B rests primarily on the decoupling of the internal spin states from any surrounding thermal reservoir. The decoupling is assured by the energy degeneracy of the spin states $|0\rangle$ and $|1\rangle$. However, in any given physical implementation, there will be limits to the accuracy with which the energy degeneracy condition could be met. For example, there may be weak residual magnetic fields in the vicinity of the spins that would lift their energy degeneracy and produce an energy cost, ϵ , associated with the CNOT operation. Nevertheless it would still be possible, in principle, to incur an energy cost much less than $kT \ln(2)$ per bit erased, where T is the temperature of the thermal reservoir, if the erasure protocol could be completed in a sufficiently short time. The time scale on which non-ideal effects become significant is determined by the coupling strengths associated with energy exchange with the surrounding reservoir, which can be small. Provided that the erasure protocol can be completed in a time much shorter than this coupling time, the internal reservoir state would be essentially independent of the temperature of the thermal reservoir. The average energy cost of the CNOT operations would then be in proportion to the average spin cost according to

$$\Delta E \approx \frac{\epsilon}{\hbar} \Delta J_z < \frac{\epsilon \ln(2)}{\hbar \gamma}, \quad (3.11)$$

neglecting the initial energy of the memory spin, which is independent of the temperature T . *The key issue is to perform the erasure protocol before the system*

reaches thermal equilibrium. It suffices to retain equilibrium only in the spin degrees of freedom. The lower bound of $kT \ln(2)$ on the energy cost, which is associated with thermal equilibrium, is thereby avoided.

4. Discussion

These results open up a range of topics for investigation. For example, the operation of Carnot “heat” engines operating with angular momentum reservoirs and generating angular momentum effort (or some other resource) instead of mechanical work. Another possibility is the use of a *combination of different types of reservoir*. For example, a Maxwell’s demon can operate on a single thermal reservoir to extract work from the reservoir. However there is an associated unmitigated cost in that the memory of the demon has to be erased to complete a cycle of operation. Bennett’s argument (1982) is to use Landauer’s erasure principle to do this, but the extracted work is more than balanced by the cost of erasure. Given the results above, we now know that the memory of the demon can be erased using an entirely different reservoir at no cost in energy. The cost instead could be in terms of spin angular momentum as illustrated in Fig. 1. In this scheme the demon’s memory is assumed to be represented by energy-degenerate spin states. In step (a) of the figure the demon has no memory and the thermal reservoir is in equilibrium. The demon traps the fastest moving molecules in the right partition of the reservoir in step (b). This allows the demon to extract work from the reservoir using a heat engine. In the last step (c) the demon’s memory is erased using a spin reservoir at a cost of spin angular momentum and the reservoir is allowed to return to equilibrium. The reservoir and the demon’s memory have completed a full cycle in terms of memory storage, however, the reservoir now has less heat and correspondingly less thermodynamic entropy. The information-theoretic entropy of the spin reservoir is nevertheless higher than at the beginning of the cycle due to the erasure of the demon’s memory. The scheme represents a cyclical process of *extracting work from a single heat reservoir* at a cost of another resource (here angular momentum) and a higher overall information-theoretic entropy.

Although this may appear to be a contentious result, it should not necessarily be regarded as contradicting various historical statements of the second law of thermodynamics within their intended contexts. For example, consider Kelvin’s dictum “It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest part of the surrounding objects” (Kelvin 1882, p179). The presence of the demon in our analysis, which in principle could be an automated machine, is not of any significance here. Rather, Kelvin’s discussions are exclusively within the context of heat and thermal reservoirs which were of overriding importance at the time of his work and, quite naturally, he did not allow for a broader class of reservoirs of the kind considered here. Our analysis therefore lies outside of Kelvin’s considerations and within a more general context. For example, our results do not appear contentious at all for an analogous, but broader, statement of the second law as: It is impossible to derive mechanical effect from any portion of matter through a reduction in the information-theoretic entropy of the system as a whole. The foregoing discussion illustrates the potential impact of a no-energy-cost erasure protocol.

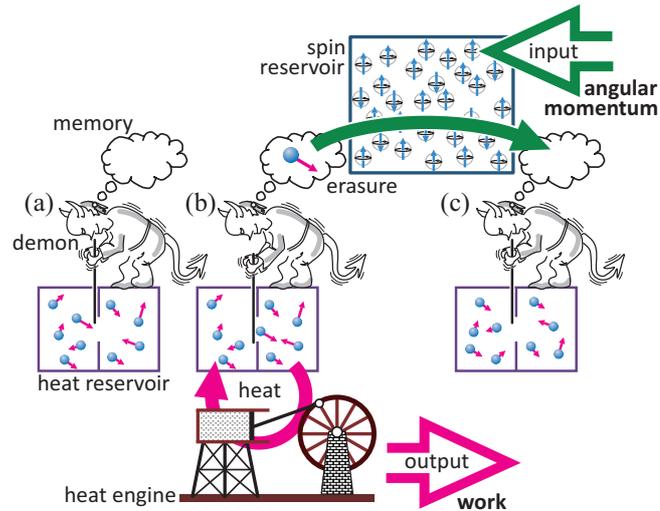


Figure 1. Maxwell's demon extracting work from a single heat reservoir at a cost of spin angular momentum. In step (a) the demon has no memory and the gas in the heat reservoir is in thermal equilibrium. Next, in step (b), the demon performs measurements of the speeds of the molecules and partitions the reservoir in two, trapping the fastest moving molecules in the partition on the right side and uses a heat engine operating between the two partitions to extract work. Finally in step (c) the demon's memory is erased using a spin reservoir and the two partitions are allowed to return to equilibrium. (Online version in colour.)

A quite different approach to this problem has been considered recently by Sagawa and Ueda (2009). They explored the possibility of reducing the information erasure cost at the expense of incurring an additional cost in the measurement process that initially stores the information. Their main result is that the erasure process can incur a cost less than kTI where I is the mutual information shared between the memory and the measured system. The reduction, however, is more than compensated by the cost of the prior measurement which ensures that the total cost of measurement and erasure is bounded below by kTI . This is in accord with Landauer's principle for the ideal case where I is also the entropy of the memory device. In contrast, our analysis is in the conventional framework where the measurement process has a zero energy cost. The total energy cost of measurement and erasure in our case is not bounded below by Landauer's energy bound. Rather we have shown that the cost of erasure can be in terms of another conserved quantity such as angular momentum.

We wish to emphasise that we have focused here on the principle of information erasure without an energy cost of the form $kT \ln(2)$. We acknowledge that the physical effort to realise these protocols would be significant especially for large scale reservoirs. But this does not lessen the conceptual significance of our results. On the contrary, by giving an explicit example, we have demonstrated that *physical laws do not forbid information erasure with a zero energy cost on principle*. Practical limitations, like the accuracy at which the degeneracy condition can be met, might well indicate that in some given physical implementation there is an unavoidable nonzero energy cost associated with erasure. But there is no fundamental reason

to suppose that this cost will necessarily be as large as $kT \ln(2)$. Moreover, the actual energy cost of erasure in such cases would depend on the particular physical implementation being considered. What is important here is the lower bound of this cost allowed by physical laws. Our results show that *the lower bound of the energy cost for the erasure of information is zero*. To this extent our results provide fresh insight into the physical nature of information.

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