Levels of Modality for BDI Logic

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Abstract

The use of rational agents for modelling real world problems has both been heavily investigated and become well accepted, with BDI (Beliefs, Desires, and Intentions) Logic being a widely used architecture to represent and reason about rational agency. However, in the real world, we often have to deal with different levels of confidence in the beliefs we hold, desires we have, and intentions that we commit to. This paper extends our previous framework that integrated qualitative levels of beliefs, desires, and intentions into BDI Logic. We describe an expanded set of axioms and properties of the extended logic. We present a modular structure for the semantics which involves a non-normal Kripke type semantics that may be used for other agent systems. Further, we demonstrate the usefulness of our framework with a scheduling task example.

Key words: Modal Logic, Agents, BDI Agents, Belief, Goals, Reasoning about belief and knowledge

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1. Introduction and Motivation

As a way of representing and reasoning about complex real world problems in the field of information and communication technologies, agent technologies are now well recognised [15] and in particular, BDI Logic is one of the most widely studied formal languages for modelling rational agents. BDI Logic originated from the early work of Bratman [7], was chiefly developed by Rao and Georgeff [17, 19], and agent languages such as AgentSpeak [18], and architectures such as JASON [5, 6] and JACK [8] are based on it. The main goal of BDI agent frameworks is to model human-like reasoning by capturing the mentalistic notions or attitudes of belief, desire, and intention. In the real world, these notions can not be simply evaluated in terms of true or false. We argue that, like humans, the agent must have an ability to reason with different levels of mentalistic notions, such as strong belief, moderate belief, weak belief, disbelief, etc. These levels of an agent's attitudes reflect the degree of its confidence about its beliefs, its desires, and its intentions and thereby allow more versatility in modelling situations.

As a simple example, let us introduce the personal assistant software of an academic (Helen). Part of the duties of this assistant software involves arranging Helen's schedule of attending seminars, etc. The system receives email notifications of seminars, meetings, etc, and using its database of Helen's beliefs, desires, and intentions, it allocates a schedule for Helen. Unfortunately, with standard BDI, there may be several seminars, meetings, or combinations of these occurring at the same time that Helen desires to attend. The system is unable to decide which to assign without some method of differentiation. We shall look further at this example later in this paper.

A basic framework was proposed in our previous work [3] to integrate 5 basic levels of grading for each of the BDI mentalistic notions and therefore give that differentiation. That framework presented a common syntax and underlying logic for the mentalistic notions, and was extended loosely from the multi-modal BDI logic of [19] with each grading or level being a separate modality. It also introduced *doxastic ignorance* (an effective absence of real belief) as one of the belief levels, and goal indifference (an absence of desire/intention) as one of the desire / intention levels. Also noted was a similarity with the framework presented by Casali *et al* in [9]. Like the framework in [3] and this paper, Casali introduces levels in all the mentalistic notions of BDI, as well as, using numeric, possiblistic type functions in its semantics. However, the similarity ends there. Casali's framework uses

multi-contexts with a different semantics for each of the mentalistic notions, though a common underlying multi-valued Lukasiewicz logic is used. This tends to make it overly complex, unwieldy and somewhat counter-intuitive. While there are nominally three modalities, they are more akin to possibility functions. Desires tend to be combined arbitrarily with no defined method for calculating the result. In this paper's second section, we will extend the basic framework syntax given in [3]. The logic's levels will be extended from that previous paper, and explained with the appropriate axioms and properties presented. The theory of having levels of belief will be first explained and further motivated. Some of the most obvious axioms and properties will be examined, including the KD45 axioms, which are pertinent to belief. This will be followed by a discussion on goals (desire and intention) and how they are incorporated into this same framework of levels. In section 3 a general non-normal Kripke type semantics, which may be used for any framework that uses levels of modality, is presented. The semantics of an agent, in particular a BDI agent, is presented and then incorporated with the Kripke semantics into an overall system. The scheduling example with the academic Helen begun earlier in the current section will be extended in section 4 using this framework to show its versatility and give an intuitive result. The paper is concluded in section 5.

2. Framework Syntax

The alphabet of this framework is the union of the following pairwise disjoint sets of symbols: a non-empty countable set $\mathcal{P}$ of atomic propositions; the set $\{\wedge, \vee, \rightarrow, \neg\}$ of connectives; the set of brackets $\{(, [,),]\}$; and a set of modalities *MOD*.

MOD is the cartesian product of the set of identifiers of the mentalistic attitudes $\{\mathrm{B}, \mathrm{D}, \mathrm{I}\}$ and the set of *Levels*, where *Levels* $= \{\mathrm{A}, \mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_n, \mathrm{E}, \mathrm{I}, \mathrm{W}_n, \mathrm{W}_{n-1}, \ldots, \mathrm{W}_1, \mathrm{D}\}$. Therefore *MOD* will be made up of such modalities as BA, DU_2, and II. The fact the D and I are elements in both the attitudes and the *Levels* should not cause confusion as their positioning in the modality will denote which they are. For more explanation on the meanings of the various modalities and levels, see the next sub-section. The syntax of the language is as follows:

$$\varphi :: \; p|(\mathrm{B}\Phi\varphi)|(\mathrm{P}\Phi\varphi)|(\neg\varphi)|(\varphi_1 \wedge \varphi_2)|(\varphi_1 \vee \varphi_2)|(\varphi_1 \rightarrow \varphi_2)$$

where $\varphi \in \mathcal{L}$ ($\mathcal{L}$ is the set of all formulae of the alphabet), $p \in \mathcal{P}$, and $\Phi \in Levels$. We write $(\varphi_1 \equiv \varphi_2)$ to abbreviate $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$.

2.1. Levels of Belief

As intimated in the introduction, in realistic situations, agents may have shades of belief. Anyone familiar with Nonmonotonic Logic, in general, and Defeasible Logic, in particular [1] [2] [16], would recognise that there are situations where something may be believed to be usually true, (e.g. any random given bird is usually able to fly), or conversely, believed usually not true (or weakly believed). There may be situations where the level of belief matches the level of disbelief (i.e. the agent doesn't know what to believe). In [3], 5 levels of belief (and desire and intention) were presented. BA was the mode of absolute belief, equivalent to the normal belief of Doxastic logic, BU was 'usual belief', BI doxastic ignorance, BW weak belief or 'usually not' believed, and BD signified disbelief. In the current work, the BU and BW levels are expanded to n levels of each, with the value of n depending on the domain (e.g. for our scheduling example presented later, n is set at 5 making 14 levels in all). Previously, Doxastic Ignorance was defined as including both indecisive belief and a lack of belief [3]. We split this into two logical terms and modalities. *Doxastic Ignorance* is a term to denote something in which no belief is held at all and is similar to the logic presented in [14], but differs in that Doxastic Ignorance naturally pertains to belief, and not knowledge. *Doxastic Equivalence* is a term to denote equal evidence for and against a belief. A description of the various levels of belief is now presented.

The belief levels we consider are defined as follows.

- BAφ means φ is believed absolutely and is the strongest belief level (e.g. "the sun will rise tomorrow").

- $\mathrm{BU}_i\varphi$ means φ is usually believed true (e.g. φ is "the bus will be on time") where $i \in \{1, 2, \ldots, n\}$ and the number of levels n within BU is domain and application dependent. So, $\mathrm{BU}_1\varphi$ means φ is believed slightly less than at BA level. Then $\mathrm{BU}_2\varphi$ means φ is believed slightly less than at BU_1 and so on down to BU_n which signifies belief slightly more than BE and perhaps BI.

- BEφ means φ is equally believed and disbelieved, that is the agent has equal evidence for and against φ. We label it as *Doxastic Equivalence*.

- $\mathrm{BI}\varphi$ means φ is neither believed nor disbelieved and is held in an absence of belief. We label it as *Doxastic Ignorance*.

- $\mathrm{BW}_i\varphi$ means φ is usually not believed, or only weakly believed, where the number of levels within BW will be the same as within BU. BW_i is the mirror opposite of BU_i so that BW_n is belief slightly less than Doxastic Eqivalence with BW_{n-1} slightly less than that and down to BW_1 which is belief slightly more than total disbelief.

- $\mathrm{BD}\varphi$ means φ is absolutely disbelieved (e.g. φ is "a comet will hit my house tonight") and is the mirror of BA.

- Doxastic possibility (P) is the $\Diamond$ (diamond) to belief's $\Box$ (box) (i.e. $\mathrm{P}\Phi\varphi = \neg\mathrm{B}\Phi\neg\varphi$). Following Hintikka [12], the reading of $\Diamond\varphi$, where the interpretation of $\Box$ is that of a particular type of belief, is "φ is consistent with what is believed". Thus, for example, $\mathrm{PU}_i\varphi$ means "φ is consistent with what is usually believed".

A formula being the *mirror* of another can be explained by example as follows. $\mathrm{BD}\varphi$ can be thought of as being equivalent to $\mathrm{BA}\neg\varphi$. So, in plain speak, disbelief in φ is the same as belief (absolute) in φ not being true, and naturally the converse is also true ($\mathrm{BA}\varphi \equiv \mathrm{BD}\neg\varphi$). This is reiterated and explained in more detail in sub-section 2.2. This leads to it being obvious that there is a natural affinity between the absolute belief level and the absolute disbelief level as well as between the levels of usual belief (BU_i) and those of usual disbelief (BW_i). This suggests the ability to cut the levels down by approximately half by eliminating BD and the BW levels. However, with only BA, the BU levels, BE, and BI, priority direction between levels could alter depending on a formula's sign. Therefore, all levels are retained here to simplify the reasoning using essentially the positive form of formulae.

2.2. Belief Axioms and Properties

In this section we present belief axioms, and properties that follow from those axioms. Belief axiom numbering is prefixed by "A_B" and properties by "P_B". Let $\varphi, \psi \in \mathcal{L}, n \in \mathbb{Z}^+, i \in \{1, 2, \ldots, n\}$, and $\Phi, \Psi, \Omega \in$ *Levels*.

$$\begin{aligned}\mathrm{BA}\varphi \vee \mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \mathrm{BI}\varphi \vee \\ \mathrm{BE}\varphi \vee \mathrm{BW}_n\varphi \vee \mathrm{BW}_{n-1}\varphi \vee \ldots \vee \mathrm{BW}_1\varphi \vee \mathrm{BD}\varphi.\end{aligned} \tag{$\mathsf{A}_\mathsf{B}1$}$$

At least one of the belief levels hold for each φ in $\mathcal{L}$. Whatever formula is selected, it must be believed at some level of belief, even if that level is only Doxastic Ignorance.

$$\text{If } \Phi \neq \Psi, \text{then } \mathrm{B}\Phi\varphi \rightarrow \neg\mathrm{B}\Psi\varphi. \tag{$\mathsf{A_B}2$}$$

For each formula in $\mathcal{L}$, at most one belief level holds.

$$\text{If } \varphi \equiv \psi, \text{then } \mathrm{B}\Phi\varphi \equiv \mathrm{B}\Phi\psi. \tag{$\mathsf{A_B}3$}$$

Belief does not depend on the syntax of the formula.

$$\mathrm{BA}\varphi \equiv \mathrm{BD}\neg\varphi. \tag{$\mathsf{A_B}4$}$$

$$\mathrm{BD}\varphi \equiv \mathrm{BA}\neg\varphi. \tag{$\mathsf{P_B}1$}$$

$$\text{For each } i \in \{1, 2, \ldots, n\}, \mathrm{BU}_i\varphi \equiv \mathrm{BW}_i\neg\varphi. \tag{$\mathsf{A_B}5$}$$

$$\text{For each } i \in \{1, 2, \ldots, n\}, \mathrm{BW}_i\varphi \equiv \mathrm{BU}_i\neg\varphi. \tag{$\mathsf{P_B}2$}$$

The belief modalities BD and BW_i are the mirror opposites of BA and BU_i respectively. For example, say φ represents the proposition *the grass in my front yard is green*, and the formula $\mathrm{BA}\varphi$ is deemed true (φ is absolutely believed). Therefore $\neg\varphi$ will mean φ is not true or effectively mean *the grass in my front yard is NOT green* and $\mathrm{BD}\neg\varphi$ (disbelieve that *the grass in my front yard is not green*) is consistent with our example belief in $\mathrm{BA}\varphi$. In fact, the two belief formulae are equivalent as indicated in axiom $\mathsf{A_B}4$. This mirroring is the same with all the other levels except BE and BI (see $\mathsf{A_B}6$ and $\mathsf{A_B}7$ for the axioms involving BE and BI). Therefore, if it is the case that the agent believes that almost always the said grass is green (e.g. $\mathrm{BU}_1\varphi$), this is equivalent to the same agent's believing very weakly (or usually not) that it is not the case that this grass is green ($\mathrm{BW}_1\neg\varphi$). Naturally the property $\mathsf{P_B}1$ follows from $\mathsf{A_B}4$ and $\mathsf{P_B}2$ similarly follows from $\mathsf{A_B}5$.

$$\mathrm{BE}\varphi \equiv \mathrm{BE}\neg\varphi. \tag{$\mathsf{A_B}6$}$$

$$\mathrm{BI}\varphi \equiv \mathrm{BI}\neg\varphi. \tag{$\mathsf{A_B}7$}$$

If a formula φ is held in *Doxastic Equivalence* it is the case that the agent cannot decide between believing for or against φ (perhaps due to equal evidence/belief for φ and $\neg\varphi$). If φ is held in *Doxastic Ignorance* it is the case that the agent has no level of actual belief at all in φ (perhaps due to

no knowledge about φ). It is therefore obvious that if φ is held in Doxastic Equivalence or Doxastic Ignorance, then $\neg\varphi$ must be similarly held in Doxastic Equivalence or Doxastic Ignorance respectively.

$$\begin{gathered}\mathrm{B}\Pi\varphi \rightarrow \neg\mathrm{BI}\varphi \wedge \neg\mathrm{BI}\neg\varphi \\ (\text{where } \Pi \in \mathit{Levels} - \{\mathrm{I}\}).\end{gathered} \tag{$\mathsf{P_B}3$}$$

It follows from $\mathsf{A_B}1$, $\mathsf{A_B}2$ and $\mathsf{A_B}7$, that if a formula is held in an actual level of belief, neither the formula, nor its negation, can be held in Doxastic Ignorance.

$$\begin{aligned}\mathrm{BI}\varphi \rightarrow & \neg\mathrm{BA}\varphi \wedge \neg\mathrm{BU}_1\varphi \wedge \neg\mathrm{BU}_2\varphi \wedge \ldots \wedge \neg\mathrm{BU}_n\varphi \wedge \neg\mathrm{BE}\varphi \wedge \\ & \neg\mathrm{BE}\neg\varphi \wedge \neg\mathrm{BU}_n\neg\varphi \wedge \ldots \wedge \neg\mathrm{BU}_2\neg\varphi \wedge \neg\mathrm{BU}_1\neg\varphi \wedge \neg\mathrm{BA}\neg\varphi.\end{aligned} \tag{$\mathsf{P_B}4$}$$

Therefore, the reverse of $\mathsf{P_B}3$ is obviously true. If a formula is held in Doxastic Ignorance, it cannot be held in a level of actual belief.

$$\begin{aligned}\mathrm{BI}\varphi \rightarrow & \neg\mathrm{BA}\varphi \wedge \neg\mathrm{BU}_1\varphi \wedge \neg\mathrm{BU}_2\varphi \wedge \ldots \wedge \neg\mathrm{BU}_n\varphi \wedge \neg\mathrm{BE}\varphi \wedge \\ & \neg\mathrm{BE}\neg\varphi \wedge \neg\mathrm{BW}_n\varphi \wedge \ldots \wedge \neg\mathrm{BW}_2\varphi \wedge \neg\mathrm{BW}_1\varphi \wedge \neg\mathrm{BD}\varphi.\end{aligned} \tag{$\mathsf{P_B}5$}$$

$\mathsf{P_B}5$ follows from $\mathsf{P_B}4$, $\mathsf{P_B}1$, and $\mathsf{P_B}2$.

The following non-numbered definition will be formally generalised later.

Definition: $\mathrm{P}\Phi\varphi = \neg\mathrm{B}\Phi\neg\varphi$ and conversely, $\mathrm{B}\Phi\varphi = \neg\mathrm{P}\Phi\neg\varphi$.

$$\mathrm{BI}\varphi \equiv \neg\mathrm{PI}\varphi. \tag{$\mathsf{P_B}6$}$$

$\mathsf{P_B}6$ follows from $\mathsf{A_B}7$ and $\mathrm{BI}\neg\varphi \equiv \neg\mathrm{PI}\neg\neg\varphi \equiv \neg\mathrm{PI}\varphi$.

Bearing in mind the relationship between the belief levels, let us look at the Doxastic Possibility of each of the belief levels.

In the following property ($\mathsf{P_B}7$), the symbol (Φ^{M}) is defined as the mirror of the level Φ, that is,

$\mathrm{A}^{\mathrm{M}} = \mathrm{D}, \mathrm{U}_i^{\mathrm{M}} = \mathrm{W}_i, \mathrm{E}^{\mathrm{M}} = \mathrm{E}, \mathrm{I}^{\mathrm{M}} = \mathrm{I}, \mathrm{W}_i^{\mathrm{M}} = \mathrm{U}_i, \mathrm{D}^{\mathrm{M}} = \mathrm{A}$; where $i \in \{1, 2, \ldots, n\}$.

$$\mathrm{P}\Phi\varphi \equiv \bigvee[\{\mathrm{B}\Psi\varphi : \Psi \in \mathit{Levels}\} - \{\mathrm{B}\Phi^{\mathrm{M}}\varphi\}]. \tag{$\mathsf{P_B}7$}$$

The property $\mathsf{P_B}7$ is deduced by expanding Definition 1 for any of the belief levels. For example, Doxastic Possibility for absolute belief of a formula is, by definition, the negation of absolute belief of the negation of that

formula. By $\mathsf{P_B1}$, this is then equivalent to the negation of disbelief of the formula. So, if the result is not disbelief, it must be one of the other levels of belief, and this is what is encapsulated in this property. As an example, the application of $\mathsf{P_B7}$ for the U_1 level of belief is as follows:

$$\begin{aligned}\mathrm{PU}_1\varphi \equiv (&\mathrm{BA}\varphi \vee \mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \\ &\mathrm{BE}\varphi \vee \mathrm{BI}\varphi \vee \mathrm{BW}_n\varphi \vee \ldots \vee \mathrm{BW}_2\varphi \vee \mathrm{BD}\varphi).\end{aligned}$$

2.2.1. KD45 Axioms

In multi-modal BDI logic, as well as in Doxastic Logic, belief has an axiom system which includes the axioms known collectively as the $\mathsf{KD45}$ system. K is a basic axiom of Kripke systems or possible world semantics and bears certain similarities to omniscience as well as modus ponens. It is defined as $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ ("if it is necessary that the rule φ implies ψ is true, then if it is necessary that φ is true, then this implies that it is necessary that ψ must be true"). D is the axiom of seriality and is defined as $\Box\varphi \to \Diamond\varphi$ ("if something is necessarily true, this implies that it is possibly true"). Axiom $\mathsf{4}$ is the axiom of transitivity and importantly, is the axiom of positive introspection. It is defined as $\Box\varphi \to \Box\Box\varphi$ ("if something is necessarily true, then it is necessary that it is necessarily true"). This is closely aligned with axiom $\mathsf{5}$ or Euclidity, the axiom of negative introspection, which is defined as $\Diamond\varphi \to \Box\Diamond\varphi$ ("if something is possibly true, then it is necessary that it is possibly true"). It has already been noted in the Introduction that the Kripke type semantics for this logic is non-normal so some aspects of $\mathsf{KD45}$ may not hold. Next we will look at what exactly does hold of $\mathsf{KD45}$ for our levels of belief.

It has been shown elsewhere [13], that the K axiom does not hold over modality gradings. An example of the K axiom for the BD level of belief,

$$\mathrm{BD}(\varphi \to \psi) \to (\mathrm{BD}\varphi \to \mathrm{BD}\psi),$$

is that if we strongly disbelieve the rule *if the sun is shining then it is raining*, then it follows that if we disbelieve *the sun is shining*, this implies that we disbelieve that *it is raining*. This is plainly counterintuitive and the other levels (except for BA) have this problem to varying degrees. However, we present here axiom $\mathsf{A_B8}$, which we derived and altered from the K axiom, and which makes much more sense.

$$\mathrm{B}\Phi(\varphi \to \psi) \to (\mathrm{BA}\varphi \to \mathrm{B}\Phi\psi). \qquad (\mathsf{A_B8})$$

$\mathsf{A_B}8$ can be more commonly written as

$$\mathrm{BA}\varphi \wedge \mathrm{B}\Phi(\varphi \to \psi) \to \mathrm{B}\Phi\psi.$$

Unfortunately, the similar rule

$$\mathrm{B}\Phi\varphi \wedge \mathrm{BA}(\varphi \to \psi) \to \mathrm{B}\Phi\psi,$$

is not valid. It would be valid if $\varphi \equiv \psi$, but as is, the most accurate possible axiom for this configuration would be

$$\mathrm{BA}(\varphi \to \psi) \to (\mathrm{B}\Phi\varphi \to \mathrm{B}\Psi\psi),$$

where Ψ is equal to, or at a higher (more believed) level than Φ. We will note this for future reference, but not make it an axiom at this stage.

The D axiom (serial) can be applied without any alteration to all levels except BE and BI and is represented as the property $\mathsf{P_B}8$.

$$\begin{gathered}\mathrm{B}\Pi\varphi \to \mathrm{P}\Pi\varphi \\ (\text{where } \Pi \in \mathit{Levels} - \{\mathrm{E},\mathrm{I}\}).\end{gathered} \tag{$\mathsf{P_B}8$}$$

Notice that this axiom corresponds to the internal consistency of the modal operator. This means that the agent is taken to be a rational agent. An example of $\mathsf{P_B}8$ for level U_1 is

$$\mathrm{BU}_1\varphi \to \mathrm{PU}_1\varphi,$$

and by using $\mathsf{P_B}7$, this can be converted to

$$\begin{aligned}\mathrm{BU}_1\varphi \to (&\mathrm{BA}\varphi \vee \mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \\ &\mathrm{BE}\varphi \vee \mathrm{BI}\varphi \vee \mathrm{BW}_n\varphi \vee \ldots \vee \mathrm{BW}_2\varphi \vee \mathrm{BD}\varphi).\end{aligned}$$

The left side of the implication, $\mathrm{BU}_1\varphi$, is included on the right side and this is then a tautology. A problem with applying D to the BI level is as follows. $\mathrm{P}\Phi\varphi = \neg\mathrm{B}\Phi\neg\varphi$ for level I is

$$\mathrm{PI}\varphi = \neg\mathrm{BI}\neg\varphi.$$

By axiom $\mathsf{A_B}7$, this can be converted to

$$\mathrm{PI}\varphi = \neg\mathrm{BI}\varphi.$$

So, it follows that the D axiom for I, $\mathrm{BI}\varphi \to \mathrm{PI}\varphi$, would be converted to

$$\mathrm{BI}\varphi \to \neg\mathrm{BI}\varphi.$$

This is obviously false and therefore $\mathrm{BI}\varphi \to \mathrm{PI}\varphi$ cannot be true, and nor can $\mathrm{BE}\varphi \to \mathrm{PE}\varphi$, for which the same applies through axiom $\mathsf{A_B6}$. This may not be as damaging as it appears. Remember that Doxastic Ignorance is an absence of any belief and Doxastic Equivalence is an absence of definite belief for or against the formula in question. Therefore, we can still say that D or seriality holds for all the levels of actual effective belief.

Normally, the axiom $\mathsf{4}$, or transitivity, is used for positive introspection. To demonstrate the problem with transitivity being used for positive introspection for these belief levels, let us apply transitivity to the D level of belief (disbelief) which results in

$$\mathrm{BD}\varphi \to \mathrm{BD}\ \mathrm{BD}\varphi.$$

This means that if the formula φ is strongly disbelieved, that disbelief is strongly disbelieved. By converting through $\mathsf{P_B1}$, we get

$$\mathrm{BA}\neg\varphi \to \mathrm{BA}\neg\mathrm{BA}\neg\varphi,$$

or by replacing $\neg\varphi$ with an equivalent formula ψ, we get

$$\mathrm{BA}\psi \to \mathrm{BA}\neg\mathrm{BA}\psi.$$

This is definitely neither the axiom of transitivity, nor that of positive introspection. However, by altering the $\mathsf{4}$ axiom of transitivity, we show that we can retain positive introspection. In axiom $\mathsf{A_B9}$, we lock the first belief level on the right side of the axiom to absolute belief, and thereby get true positive introspection, which is what we are really seeking here.

$$\mathrm{B}\Phi\varphi \to \mathrm{BA}\ \mathrm{B}\Phi\varphi. \tag{$\mathsf{A_B9}$}$$

The axiom $\mathsf{5}$, or Euclidean, is used for negative introspection. To demonstrate the problem with euclidity for these belief levels, let us apply the $\mathsf{5}$ axiom to the D level of belief,

$$\mathrm{PD}\varphi \to \mathrm{BD}\ \mathrm{PD}\varphi.$$

Using $\mathsf{P_B7}$, we can convert this to

$$\begin{aligned}&(\mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \mathrm{BE}\varphi \vee \mathrm{BI}\varphi \vee \\ &\qquad\qquad \mathrm{BW}_n\varphi \vee \mathrm{BW}_{n-1}\varphi \vee \ldots \vee \mathrm{BW}_1\varphi \vee \mathrm{BD}\varphi) \rightarrow \\ &\mathrm{BD}(\mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \mathrm{BE}\varphi \vee \mathrm{BI}\varphi \vee \\ &\qquad\qquad \mathrm{BW}_n\varphi \vee \mathrm{BW}_{n-1}\varphi \vee \ldots \vee \mathrm{BW}_1\varphi \vee \mathrm{BD}\varphi).\end{aligned}$$

So, if the disjunction of formulae on the left of the statement is true, then this implies that we disbelieve that same disjunction of formulae. This is not what we want and is not true negative introspection. However, we can alter the $\mathsf{5}$ axiom in a similar manner to that of our alteration of the $\mathsf{4}$ axiom to allow negative introspection. In axiom $\mathsf{A_B10}$, we again lock the first belief level on the right side of the axiom to absolute belief, and thereby get true negative introspection, which is what we really want.

$$\mathrm{P}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{P}\Phi\varphi. \tag{$\mathsf{A_B10}$}$$

These axioms of introspection hold and are perfectly consistent. As well, these altered $\mathsf{KD45}$ axioms seem to hold for all belief levels. The only possible problem is regarding the BI and BE levels, in not holding for the D axiom. However, due to BI not being an actual level of belief and BE not being a definite level of belief, this problem can be essentially ignored. So, while this logic does not strictly follow the $\mathsf{KD45}$ system because of the issues with grading, it follows the closest possible approximation of it, as we outlined.

Notice that given the combination of Axiom $\mathsf{A_B1}$ and $\mathsf{A_B2}$ we have that $\Box\top$ holds only for one belief level. Thus, we assume the following axiom.

$$\mathrm{BA}(\varphi \rightarrow \varphi). \tag{$\mathsf{A_B11}$}$$

Accordingly, we have that modal operator BA is a normal modal operator, while all other belief levels are characterised by non-normal modal operators.

2.3. Levels of Goals

Desires and Intentions can both be considered types of goals, with desires being weak goals and intentions being strong goals (i.e. desire + commitment = intention). As stated in the introduction, our conventional BDI agent either has a given desire, or it doesn't. But, it is quite conceivable for an agent to have desires of varying degrees of strength. For example, the desire to live/survive is going to be stronger than the desire to go to work, which in

turn is stronger than the desire to go jogging on a hot day (at least it is for the authors). So, having levels of desire gives our agent more versatility in representing a wider range of situations. Where belief of a formula φ relates mainly to φ being true in the current state of applicable possible worlds, the desire (and intention) of φ usually pertains to its truth in other (i.e. future) worlds.

The framework of levels introduced for beliefs is easily extended to desires and also intentions. Essentially, each goal level (desire/intention) will have a not dissimilar meaning to equivalent belief levels. The difference between goals being that a given desire, among several desires, will only become an intention if it is committed to by the agent. The agent may have many conflicting desires but should have no conflicting intentions if it is a rational agent.

The desire levels we consider are defined as follows (as with belief, $i \in \{1, 2, \ldots, n\}$, and $n \in \mathbb{Z}^+$):

- $\mathrm{DA}\varphi$ denotes that φ is absolutely or strongly desired and is the strongest desire level (e.g. to survive, to be happy).

- $\mathrm{DU}_i\varphi$ means φ is desired at a moderate level, less than DA, whereas with the belief U_i level, the number of levels n within DU is domain and application dependent. For example, the desire to "earn money" may be a DU_1 desire whereas the desire to "do work" may be at a weaker level of desire, say DU_3.

- $\mathrm{DE}\varphi$ applies if the formulae φ and $\neg\varphi$ are equally desired by the agent. It may apply to alternatives of which the agent has no preference of one over the other (e.g. to get to work the agent desires to "walk" / "drive" / "catch a bus"). This can be described as *Goal Indifference*.

- $\mathrm{DI}\varphi$ applies if neither φ nor $\neg\varphi$ is desired at all by the agent and could possibly also be described as goal indifference, but is probably more accurately described as *Goal Ignorance*.

- $\mathrm{DW}_i\varphi$ denotes that φ is only weakly desired, that is, a goal that the agent usually doesn't wish for (e.g. "go bungee jumping" [indicated as a DW level due to author's fear of heights - it may be a DU level for a thrill-seeking type agent]). This is the mirror of DU_i (equivalent to $\mathrm{DU}_i\neg\varphi$).

- $\mathrm{DD}\varphi$ means that φ is definitely not desired (e.g. "have an accident driving home"), and is the mirror of DA, or equivalent to $\mathrm{DA}\neg\varphi$.
- $\mathrm{P_D}$ represents desire possibility and is the diamond ($\Diamond$) to desire's box ($\Box$) ($\mathrm{P_D}\Phi\varphi = \neg\mathrm{D}\Phi\neg\varphi$).
- $\mathrm{P_I}$ represents intention possibility and is the diamond ($\Diamond$) to intention's box ($\Box$) ($\mathrm{P_I}\Phi\varphi = \neg\mathrm{I}\Phi\neg\varphi$).

As to the Intention levels, they have essentially the same meaning as the desire levels, as they are a committed desire. After deliberation, an agent commits to a particular desire, thereby creating an intention of the same level (e.g. $\mathrm{DU}_2\varphi$ + commitment = $\mathrm{IU}_2\varphi$). The desire and intention axioms and properties are essentially the same as those of belief, with the letter B being replaced by D or I respectively and P being replaced by $\mathrm{P_D}$ or $\mathrm{P_I}$ respectively. Therefore we will not take up space here stating them. Naturally there are no direct goal equivalents to the belief introspection axioms, A9 and A10. However, there may be further altered introspection axioms of the form of

$$\mathrm{D}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{D}\Phi\varphi,$$

so that if a formula is desired at a given level of desire, then the desire of that formula at that given level is absolutely believed by the agent. This would naturally also apply to intention, that is,

$$\mathrm{I}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{I}\Phi\varphi.$$

A similar extension to A10 for negative introspection of desire would be expressed by

$$\mathrm{P_D}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{P_D}\Phi\varphi,$$

and similarly for intention,

$$\mathrm{P_I}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{P_I}\Phi\varphi.$$

Again, while the semantics support these, we will not make a formal axiom of them, but merely note them for possible future use.

3. Semantics

The semantics of this logic was originally modelled loosely on the semantics of Rao & Georgeff's original logic [17, 19] in the previous work of [3, 4], but with the revisions in this paper, it has been modularised to separate those components that are independent of the agent itself. This has been carried out in a manner not dissimilar to that presented in [20, 21]. We will define a *Model* $\mathcal{M}$ to be a non-normal Kripke structure within which we might place any structure that requires positioning in a 'partitioned' worlds environment. This structure has been influenced by, and follows the general forms of, Minimal / Neighbourhood semantics [11]. We will also define an *Agent* $\mathcal{A}$, in particular a BDI agent, which encapsulates levels in its composition. Finally we will tie these two together in a *System* $\mathcal{S}$. Now we formally define each of these structures.

3.1. Neighbourhood Semantics

We introduce the neighbourhood semantics characterising the axiomatisation proposed in Section 2.2.

Definition 2: A *Model* $\mathcal{M}$ is a tuple $\langle W, N, V \rangle$, where

1) W is a set of worlds.
2) N, is a set of 'neighborhood' functions where each function $N_Z \in N$ has the signature $N_Z : W \times Levels \rightarrow 2^{2^W}$.
3) V is a function mapping atomic formulae at worlds to a truth value; that is, $V : \mathcal{P} \times W \rightarrow \{false, true\}$. We say that p is true at w iff $V(p, w) = true$; and say that p is false at w iff $V(p, w) = false$.

The above definition is essentially the standard definition of a neighbourhood model for a multi-modal logic, where each modal operator Z has its own neighbourhood function N_Z, however, in the definition we have combined all such function into a single function indexed by the modal operators (levels). As we will see the modal operators will partition the proposition in the language, thus having a single neighbourhood function simply the representation of the condition that the neighbourhood functions for the level partition the power set of possible worlds.

Definition 3: The truth of formulae φ in $\mathcal{L}$ is defined as:

1) If $\mathcal{M} = \langle W, N, V \rangle$, $p \in \mathcal{P}$, and $w \in W$, then we define
$(\mathcal{M}, w) \vDash p$ iff $V(p, w) = true$.

2) If $\mathcal{M} = \langle W, N, V \rangle$, for each modal operator, Z, there is a function $N_Z \in N$, and if $\Phi \in Levels$, $w \in W$, and $\varphi \in \mathcal{L}$, the truth of modal formulae are defined by:

$$(\mathcal{M},\, w) \vDash \mathrm{Z}\Phi\varphi \text{ iff } \|\varphi\| \in N_Z(w, \Phi)$$
$$\text{where } \|\varphi\| = \{w \in W : (\mathcal{M},\, w) \vDash \varphi\}.$$

3) If $\mathcal{M} = \langle W, N, V \rangle$, $w \in W$, and $\varphi \in \mathcal{L}$, then we define

$$(\mathcal{M},\, w) \vDash \neg\varphi \text{ iff } (\mathcal{M},\, w) \nvDash \varphi,$$
$$(\mathcal{M},\, w) \vDash \varphi \wedge \psi \text{ iff } (\mathcal{M},\, w) \vDash \varphi \text{ and } (\mathcal{M},\, w) \vDash \psi,$$
$$(\mathcal{M},\, w) \vDash \varphi \vee \psi \text{ iff } (\mathcal{M},\, w) \vDash \varphi \text{ or } (\mathcal{M},\, w) \vDash \psi,$$
$$(\mathcal{M},\, w) \vDash \varphi \rightarrow \psi \text{ iff } (\mathcal{M},\, w) \nvDash \varphi \text{ or } (\mathcal{M},\, w) \vDash \psi.$$

Now that $(\mathcal{M},\, w) \vDash \varphi$ for an arbitrary formula φ is defined, we have the following formal definition of a *truth set*.

Definition 4: If φ is any formula then the *truth set* of φ, $\|\varphi\|$, is defined by

$$\|\varphi\| = \{w \in W : (\mathcal{M},\, w) \vDash \varphi\}.$$

Definition 5: We now further refine N in the model $\mathcal{M}$ so that $\mathcal{M} = \langle W, \{N_B, N_D, N_I\}, V \rangle$. If $N_Z \in N, \Phi \in Levels$, and $w, w' \in W$, then N satisfies the following conditions:

a) If $X \in N_Z(w, \Phi)$ then $W – X \in N_Z(w, \Phi^{\mathrm{M}})$.
b) If $X \in N_Z(w, \mathrm{A})$ and $((W – X) \cup Y) \in N_Z(w, \Phi)$, then $Y \in N_Z(w, \Phi)$.
c) If $\Phi \neq \Psi$ then $N_Z(w, \Phi) \cap N_Z(w, \Psi) = \{\}$.
d) $\bigcup\{N_Z(w, \Phi) : \Phi \in Levels\} = 2^W$ for every world w.
e) If $X \in N_Z(w, \Phi)$ then $\{w' : X \in N_Z(w', \Phi)\} \in N_B(w, \mathrm{A})$.
f) If $X \notin N_Z(w, \Phi)$ then $\{w' : X \notin N_Z(w', \Phi)\} \in N_B(w, \mathrm{A})$.
g) $W \in N_Z(w, \mathrm{A})$.

Let us quickly comment on the conditions given in Definition 5. Condition a) establishes the 'mirror' relationships between the W_i and U_i modalities, thus it corresponds to axioms A4–A7. Condition b) characterises axiom A8. Condition c) states that any two distinct levels have non overlapping neighbourhood sets, thus any proposition is true (at most) for only one level; this is encoded in axiom A2; in combination with condition d) it can be seen that $\{N_Z(w, \Phi) : \Phi \in Levels\}$ is a partition of 2^W, thus we have axiom A1. Condition g) is the standard condition on neighbourhood models for axiom A11. Similarly, parts e) and f) are the condition for positive and negative introspection for a modal operator Z with respect to absolute belief.

Theorem 1: The logic axiomatised by Axioms A1–A11 is determined by the class of minimal models satisfying Definition 5.

The proof of the Theorem is given in Appendices A and B.

3.2. Agents

In the previous section we have shown how to give a possible world semantics for the logic we defined in Section 2.2. In this section we are going to show a possible way to ground an agent to the semantics we have just provided. The procedure is as follows: an agent specifies a degree of belief, desire, intention for each proposition the agent believes, desires or intends. The degree is express as a value in the range $[0, 1]$, the assignment of values is constrained by the conditions given in Definition 7. Furthermore, an agent establishes a (total) linear order (over the given range) to be able to classify the proposition over the levels of the appropriate mental attitudes (Definition 6). The final step is to build the model based on the assignments just described (Definition 8).

Definition 6:

1) For each $i \in \{1, \ldots, n\}$, we define d_i, the division points for the levels, as follows: $d_i \in \{x \in \mathbb{Q} : 0{\cdot}5 \leq x < 1\}$, $d_n = 0{\cdot}5$, and for all $i \in \{2, \ldots, n\}$, $d_i < d_{i-1}$. ($\mathbb{Q}$ represents the set of rational numbers.)
2) To relate the division points to the U and W levels, we define
   (a) $\mathbb{Q}(\mathrm{U}_1) = \{x \in \mathbb{Q} : d_1 < x < 1\}$,
   (b) for all $i \in \{2, \ldots, n\}$, $\mathbb{Q}(\mathrm{U}_i) = \{x \in \mathbb{Q} : d_i < x \leq d_{i-1}\}$,
   (c) for all $i \in \{2, \ldots, n\}$, $\mathbb{Q}(\mathrm{W}_i) = \{x \in \mathbb{Q} : 1{-}d_{i-1} \leq x < 1{-}d_i\}$, and
   (d) $\mathbb{Q}(\mathrm{W}_1) = \{x \in \mathbb{Q} : 0 < x < 1{-}d_1\}$.
3) We define the equivalence relation $\approx$ on $\mathbb{Q} \cup \{\mathit{u}\}$, where u represents *undefined*.

$$x \approx y \text{ iff either (i) } x = y, \text{ or}$$
$$\text{(ii) for some } i \in \{1, \ldots, n\}, \{x, y\} \subseteq \mathbb{Q}(\mathrm{U}_i), \text{ or}$$
$$\text{(iii) for some } i \in \{1, \ldots, n\}, \{x, y\} \subseteq \mathbb{Q}(\mathrm{W}_i).$$

Definition 7: An *Agent* $\mathcal{A}$ is a tuple $\langle \beta, \delta, \iota \rangle$ where $\beta, \delta,$ and ι are functions for belief, desire and intention respectively which take a world w and a formula $\varphi \in \mathcal{L}$ believed (resp. desired, intended) at a particular level of belief (resp. desire, intention) and returns a number that represents the level of belief (resp. desire, intention) that φ is held in at w. Suppose $\gamma \in \{\beta, \delta, \iota\}$ and $\varphi, \psi \in \mathcal{L}$, then we require:

a) $\gamma : W \times \mathcal{L} \to \{x \in \mathbb{Q} : 0 \leq x \leq 1\} \cup \{u\}$.
b) $\gamma(w, \varphi) = 1 - \gamma(w, \neg\varphi)$, and $\gamma(w, \varphi) = u$ iff $\gamma(w, \neg\varphi) = u$.
c) $\gamma(w, (\varphi \to \varphi)) = 1$, and if $\gamma(w, \varphi) = 1$ then $\gamma(w, \psi) = \gamma(w, (\neg\varphi \vee \psi))$.
d) $\beta(w, \{w' \in W : \gamma(w', \varphi) \approx \gamma(w, \varphi)\}) = 1$.

The main idea of the above definition is that an agent associates to each possible world and each proposition a level of belief, desire and intention (clause a). Then clause b) checks that the complement of a proposition is associated to the mirror level associated to the proposition. Condition c) ensures that propositions are closed under classical propositional logic. Finally, the intuition behind the last condition is fully aware (absolutely believes) to what degree she believes, intends and desires each proposition. In other terms that the agent's absolute beliefs are fully introspective.

Definition 8: We define a *System* $\mathcal{S}$ as the tuple $\langle \mathcal{M}, \mathcal{A} \rangle$ where $\mathcal{M}$ and $\mathcal{A}$ are as defined in Definition 2 and Definition 7 respectively. $\mathcal{M}$ is refined as in Definition 5, that is, N is restricted to the set of three functions N_B, N_D, and N_I. If $Z \in \{B, D, I\}$, then define Z' by $B' = \beta$, $D' = \delta, I' = \iota$. Given $\varphi \in \mathcal{L}$, we define $N_Z(w, \Phi)$ as follows:
a) $N_Z(w, \mathrm{A}) = \{\|\varphi\| \subseteq W : Z'(w, \varphi) = 1\}$.
b) For $i \in \{1, \ldots, n\}$, $N_Z(w, \mathrm{U}_i) = \{\|\varphi\| \subseteq W : Z'(w, \varphi) \in \mathbb{Q}(\mathrm{U}_i)\}$.
c) $N_Z(w, \mathrm{E}) = \{\|\varphi\| \subseteq W : Z'(w, \varphi) = 0{\cdot}5\}$
d) For $i \in \{1, \ldots, n\}$, $N_Z(w, \mathrm{W}_i) = \{\|\varphi\| \subseteq W : Z'(w, \varphi) \in \mathbb{Q}(\mathrm{W}_i)\}$.
e) $N_Z(w, \mathrm{D}) = \{\|\varphi\| \subseteq W : Z'(w, \varphi) = 0\}$.
f) $N_Z(w, \mathrm{I}) = \{\|\varphi\| \subseteq W : Z'(w, \varphi) = u\}$.

Definition 8 shows how to build a neighbourhood model starting from the degree of belief, desire and intention associated to each proposition.

A System and Agent created from the conditions defined in Definitions 6 to 8 is consistent with the conditions of a Model as defined in Definitions 2 to 5. Verification of this can be seen in Appendix C.

Reasoning among various mentalistic levels is accomplished through priorities or a total order between levels. Note that the place in this order of the level I (with a γ value of u) is domain dependent. It may, most commonly, be placed equal with E at $0{\cdot}5$. For belief, this order of levels would then effectively be

$$\mathrm{BA} > \mathrm{BU}_1 > \mathrm{BU}_2 > \ldots > \mathrm{BU}_n > \mathrm{BE} = \mathrm{BI} > \mathrm{BW}_n > \mathrm{BW}_{n-1} > \ldots > \mathrm{BW}_1 > \mathrm{BD},$$

and given $\Phi, \Psi \in Levels$,

$$\mathrm{B}\Phi\varphi > \mathrm{B}\Psi\psi$$

signifies $\mathrm{B}\Phi\varphi$ has a higher belief than $\mathrm{B}\Psi\psi$. So, $\mathrm{B}\Phi\varphi > \mathrm{B}\Psi\psi$ means that

$$\|\varphi\| \in N_B(w, \Phi),$$
$$\|\psi\| \in N_B(w, \Psi),$$

and

$$\beta(w, \|\varphi\|) > \beta(w, \|\psi\|).$$

This latter may be intuitively written as

$$\beta(\varphi) > \beta(\psi),$$

as $\beta(w, X)$ may be simplified to $\beta(\varphi)$ where the current world w is obvious and $X = \|\varphi\|$. Reasoning among desires is carried out in a similar manner. Intentions would be created by the agent committing to the desire (selected from among desires whose time for possible execution is at the same time) with the highest δ value. The ι value of that intention would nominally be set as being the same as the δ value of the desire it is created to carry out.

4. Scheduling Example

Continuing the scheduling agent example of Helen's personal assistant software started in the introduction, we examine four seminars and one meeting, notifications that Helen has received pertaining to one day. These are listed in the knowledge (facts) in Table 1. Helen has previously recorded her beliefs and desires with the scheduling system and her desires and beliefs relevant for this example are listed in Table 2 and Table 3 respectively. As can be seen, there are conflicts in the timing of the various events, thus an agent cannot attend all of them. For this example we are setting the value of n (number of U_i and W_i levels) to 5, giving 14 levels in each of belief, desire, or intention.

We now suggest rules for combining desires in the scheduling example domain. They may also be used in other domains if suited. First, we suggest that there are two types of desires, dependent and independent. Dependent desires are those that are part of a more complex desire. A desire to hear a particular speaker, or, a desire to hear a talk on a particular topic are dependent desires, that is, they need to be combined to stand alone as an

Knowledge (Facts)	**Formula**
Fred speaks on Robotics in room N$rm1$ at time t_1.	$seminar$(Fred,Robotics,N$rm1$,t_1)
Alice speaks on Agent Technology in room N$rm1$ at time t_2	$seminar$(Alice, Agents, N$rm1$, t_2)
The University Council is meeting in room L$rm1$ at time t_2.	$meeting$(ucm, L$rm1$, t_2)
Sam speaks on BDI Agents at room UQ$rm4$ at time t_2.	$seminar$(Sam, BDI, UQ$rm4$, t_2)
Rex speaks on Nonmonotonic Reasoning in room L$rm2$ at t_1.	$seminar$(Rex, NMR, L$rm2$, t_1)

Table 1: Facts relevant to the scheduling example.

Helen's Desires	δ	**Formula**
strongly to attend seminars by Fred.	·92	DU_1($semSpeak$(Fred))
to usually attend seminars by Sam.	·75	DU_3($semSpeak$(Sam))
to usually attend seminars by Alice.	·75	DU_3($semSpeak$(Alice))
to usually not attend seminars by Rex.	·25	DW_3($semSpeak$(Rex))
absolutely to attend Albert's seminars.	1	DA($semSpeak$(Albert))
indifferent to attending Robotics seminars.	·5	DE($semTopic$(Robotics))
to usually attend seminars on Nonmonotonic Reasoning.	·75	DU_3($semTopic$(NMR))
to usually attend seminars on Agent Technology.	·75	DU_3($semTopic$(Agents))
strongly to attend BDI Agent seminars.	·95	DU_1($semTopic$(BDI))
strongly to *not* attend IS seminars.	·12	DW_2($semTopic$(InfoSys))
never to attend Image Processing seminars.	0	DD($semTopic$(ImgProc))
to usually attend University Council Meetings	·75	DU_3($attend$(UCM))

Table 2: Helen's desires including those relevant to scheduling example.

independent desire. For example, we desire to hear Fred speak, but he will be speaking on a particular topic. We are indifferent in our desire to hear the

Helen's Beliefs	β	**Formula**
believes strongly that N$rm1$ is easy to travel to.	·97	$\mathrm{BU}_1(ease[\mathrm{goto}(\mathrm{N}rm1)])$
believes that location L$rm1$ is moderately easy to travel to.	·85	$\mathrm{BU}_2(ease[\mathrm{goto}(\mathrm{L}rm1)])$
believes that location L$rm2$ is moderately easy to travel to.	·85	$\mathrm{BU}_2(ease[\mathrm{goto}(\mathrm{L}rm2)])$
believes absolutely the wait is short.	1	$\mathrm{BA}(ease[\mathrm{wait}])$
believes that UQ$rm4$ is usually easy to travel to.	·75	$\mathrm{BU}_3(ease[\mathrm{goto}(\mathrm{UQ}rm4)])$
believes that location L$rm1$ is very easy to travel to from L$rm2$.	·97	$\mathrm{BU}_1(ease[\mathrm{gofromto}(\mathrm{Ł}rm2, \mathrm{L}rm1)])$

Table 3: Helen's beliefs relevant to scheduling example.

topic robotics, so if Fred is speaking on robotics, we suggest that the desire to hear Fred talk on Robotics is less than the desire to hear Fred speak, but more than the desire to hear about Robotics. Formally, to find the δ value of two dependent desires a and b (of the form $\mathrm{D}\Phi\varphi$ where $\Phi \in Levels$ and $\varphi \in \mathcal{L}$ [note that we abbreviate $\delta(a)$ to δa and $\delta(b)$ to δb]), we use the formula:

$$\delta(a \wedge b) = (\delta a + \delta b) \div 2 \tag{F1}$$

Combining two independent or stand alone desires, is totally different, e.g. the desire to attend two seminars should be higher than the desire to only attend either one of them, all other issues being equal. To find the δ value of two independent desires a and b (of the form $\mathrm{D}\Phi\varphi$), we use the formula:

$$\delta(a \wedge b) = \begin{cases} f(\delta a, \delta b) & \text{if } \delta a > 0{\cdot}5 \text{ and } \delta b > 0{\cdot}5 \\ 1 - f(1 - \delta a, 1 - \delta b) & \text{if } \delta a < 0{\cdot}5 \text{ and } \delta b < 0{\cdot}5 \\ \delta a + \delta b - 0{\cdot}5 & \text{otherwise} \end{cases} \tag{F2}$$

The formula F2 agrees with Casali's unspecified approach in her travel example [9, 10].

An approximate graphical representation of the relevant worlds with their desires, beliefs and intentions is presented in Figure 1. This diagram includes the separate 'seminar' events as possible worlds, with the world w_6 containing the two events represented in worlds w_1 and w_2. This is to allow the

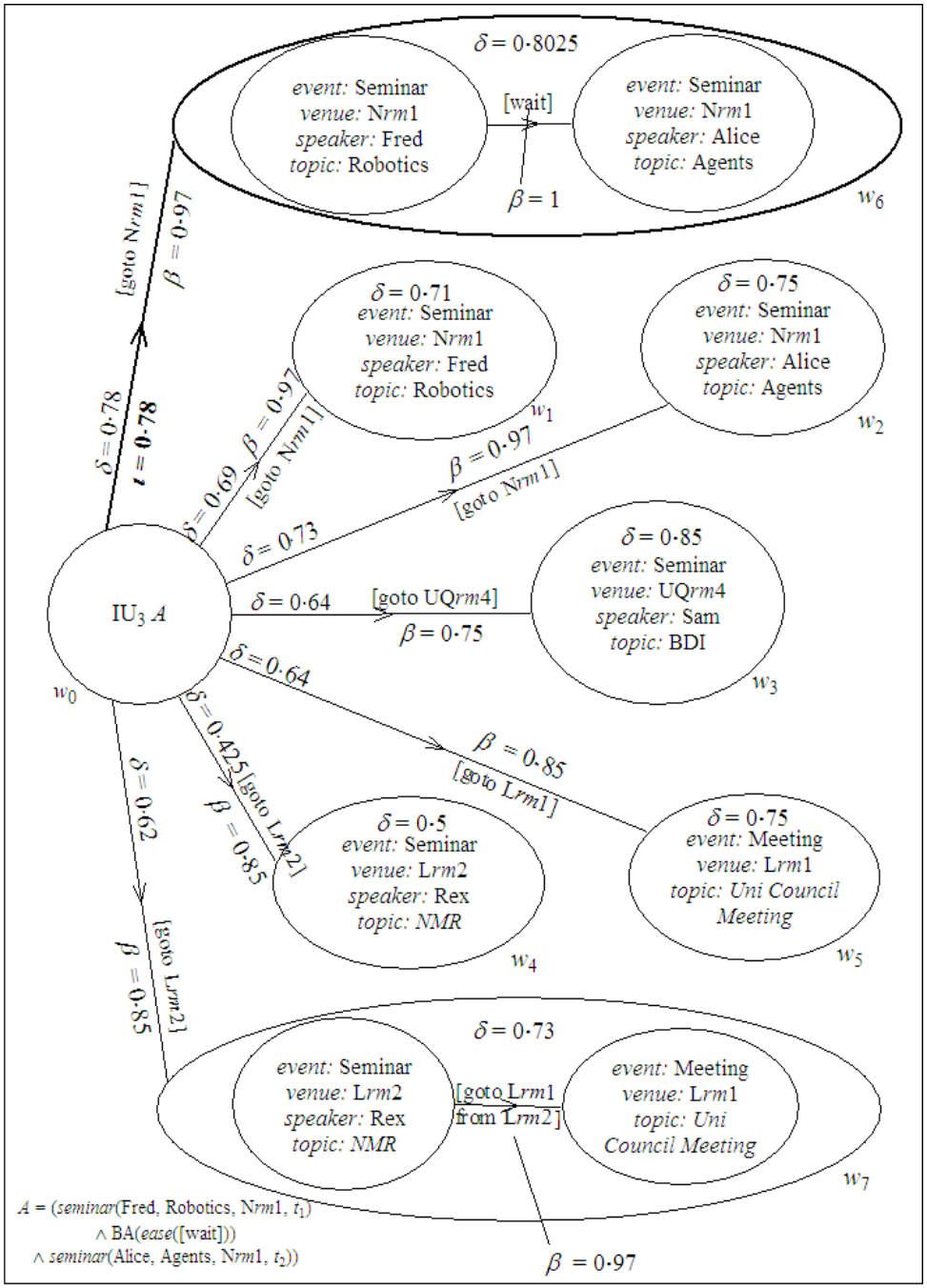

Figure 1: Collected belief, desire and intention worlds relevant to Tables 1 to 3 from $N_B(w_0), N_D(w_0)$, and $N_I(w_0)$, (where $N_Z(w_0) = \bigcup\{N_Z(w_0, \Phi) : \Phi \in Levels\}$).

consideration of the desire of attending w_1, or w_2, or attending both of them together. Also w_7 contains the two events represented in worlds w_4 and w_5 so that they can be similarly considered.

Therefore, evaluating the basic desire of *seminar*(Fred, Robotics, N$th1$, t_1) (i.e. w_1 in Figure 1) using F1 is:

$$\delta(\mathrm{DU}_1(\mathit{semSpeak}(\mathrm{Fred})) \wedge \mathrm{DE}(\mathit{semTopic}(\mathrm{Robotics}))) = ((\cdot 92 + \cdot 5) \div 2) = \cdot 71.$$

So, while the above gave us the basic desire for world w_1 has a δ value of $\cdot 71$, to get the overall δ value of *attending* w_1, we must apply the belief to getting to the associated event. To get this we apply the product of the belief to that basic desire to result in the end desire.

$$\begin{aligned}
&\delta(\beta(\mathit{ease}[\text{goto N}rm1]) \wedge \delta(\mathit{seminar}(\text{Fred, Robotics, N}th1, t_1))) \\
&\quad = \cdot 97 \times \cdot 71 \\
&\quad = \cdot 6887 \text{ (rounded to } \cdot 69).
\end{aligned}$$

Similarly, *seminar*(Alice, Agents, N$rm1$, t_2) (i.e. w_2) results in a δ value of $\cdot 75$ and after applying the belief about getting there, becomes a value of $\cdot 73$. Therefore, the evaluation of the desire of world w_6 which combines both the seminars of w_1 and w_2, is calculated via F2 as:

$$\begin{aligned}
&\delta(\delta(\mathit{seminar}(\text{Fred, Robotics, N}rm1, t_1)) \wedge \\
&\qquad\qquad (\beta(\mathit{ease}[\text{wait}]) \wedge \delta(\mathit{seminar}(\text{Alice, Agents, N}rm1, t_2)))) \\
&= \delta(\cdot 71 \wedge (1 \times \cdot 75)) \qquad\qquad (\textit{waiting and seminars' desires combined}) \\
&= max(\cdot 71, \cdot 75) + ((1 - max(\cdot 71, \cdot 75) \times min(\cdot 71, \cdot 75) - \cdot 5)) \\
&= \cdot 75 + ((1 - \cdot 75) \times (\cdot 71 - \cdot 5)) \\
&= \cdot 75 + (\cdot 25 \times \cdot 21) \\
&= \cdot 75 + \cdot 0525 \\
&= \cdot 8025.
\end{aligned}$$

Note the overall δ value of the second event is different from that of w_2 because of the difference of the belief of merely waiting at N$rm1$ as opposed to getting to N$rm1$ in w_2. For the overall δ value of w_6, we apply the belief value of getting to N$rm1$: $\cdot 8025 \cdot 97 = \cdot 7784$ (rounded to $\cdot 78$).

By similar calculations, attending *seminar*(Sam, BDI, UQ$rm4, t_2$)(w_3) has an overall δ value of $\cdot 85 \times \cdot 75 = \cdot 6375$ ($\cdot 64$), attending *seminar*(Rex,

NMR, $\mathrm{L}rm2, t_1)(w_4)$ has an overall δ value of $\cdot 5 \times \cdot 85 = \cdot 425$, and *meeting*(ucm, $\mathrm{L}rm1, t_2)(w_5)$ has an overall δ value of $\cdot 75 \times \cdot 85 = \cdot 6375$ ($\cdot 64$). So, by F2, the overall δ value of w_7 is:

$$
\begin{aligned}
&\delta(\beta(\mathit{ease}[\text{goto L}rm1]) \wedge (\delta(\mathit{seminar}(\text{Rex, NMR, L}rm2, t_1))\wedge \\
&\quad (\beta(\mathit{ease}[\text{gofromto L}rm2 \text{ from L}rm2]) \wedge \delta(\mathit{meeting}(\text{ucm, L}rm1, t_2))))) \\
&= \cdot 85 \times (\cdot 5 + (\cdot 97 \times \cdot 75) - \cdot 5) \\
&= \cdot 85 \times (\cdot 5 + \cdot 7275 - \cdot 5) \\
&= \cdot 85 \times \cdot 7275 \\
&= \cdot 6184 \qquad\qquad (\text{rounded to } \cdot 62).
\end{aligned}
$$

Note that the desire calculation of w_7, did not use the function f within F2, but calculation went to the 'otherwise' of $(\delta a + \delta b - 0 \cdot 5)$ because δa was $0 \cdot 5$. This really justifies this aspect of F2, as the calculation using f would have been $0 \cdot 86$ (before applying the belief of ease of going there). Logically, a desire of indifference should not boost (or drop) another desire when combined with it. In the calculation used, before the applying of the belief of going to the pair of seminars, the δ was $\cdot 7275$, exactly the δ value of the other desire (being the combination of the meeting and going to the meeting). Therefore, the desire with a δ of $0 \cdot 5$ had a neutral effect when combined with another desire, which is as it should be.

So the world w_6 results in being the most desired world ($\cdot 78$ over $\cdot 73, \cdot 69, \cdot 64, \cdot 62$ and $\cdot 425$) and the desire to bring about w_6 is therefore converted to an intention with an ι value of $\cdot 78$. This equates to the relevant formulae in w_6 being intended at an intention level of IU_3.

5. Conclusion and Future Work

The framework undertaken in this paper provides an environment that may be used with any agent or device that requires graduated modality. This is extended to give a foundation for a layered BDI architecture, which essentially enables a rational agent to capture commonsense reasoning. We believe that representing and reasoning with levels of mentalistic attitudes significantly enhances an agent's ability to perform human-like practical reasoning in complex domains. The proposed framework is simpler and more intuitive than other BDI frameworks, including that of Casali. Intended future work involves introducing this basic framework into an existing BDI agent platform, most likely the AgentSpeak(L) platform, JASON. Naturally

this may necessitate dropping the strictly modal aspects of this logic, but the semantics, as stated, should be able to be easily adapted to JASON.

Some issues left unanswered by this work are that often it is not clear how many levels of the mentalistic notions are needed in a given application, and how to determine the division points, and the degree of belief, intention and desire. In the scheduling example we have used adjectives qualifying the various mentalistic notions (e.g., "strongly", "usually", "strongly not",...). A similar approach can be used to describe the levels in a particular application based on the knowledge of the domain by a domain expert. For the second and third issues we have used quantitative methods, but we could have used qualitative methods where the designer of an application can simply elicit the information again using adjectives to qualify the various levels (provided the adjectives determine an order of the levels).

This paper was part of Jeff's research for his PhD. This paper constitutes part of the theoretical foundation of how to model agents. Jeff passed away before he was able to complete his research. He was working on methodologies to build agent based applications and he was investigating some of the issues discussed above.

Appendices

A. Soundness Proofs

In this appendix, we shall prove that each of the axioms and properties given in section 2.2 are sound in respect of the semantics given in section 3.

A.1. Assistant Lemmas

First we give and prove some lemmas that will be helpful in our soundness proofs.

Let φ and ψ be any formulae in our countable language $\mathcal{L}, n \in \mathbb{Z}^+$, $\Phi, \Psi \in \mathit{Levels}$, and $\mathcal{M} = \langle W, N, V \rangle$ be a model.

Lemma A1.1: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model.
If $\forall w \in W\ (\mathcal{M}, w) \vDash (\varphi \equiv \psi)$, then $\|\varphi\| = \|\psi\|$.

Proof:
Suppose that $\forall w \in W$,
$(\mathcal{M}, w) \vDash (\varphi \equiv \psi)$.
Take any $w' \in \|\varphi\|$, and it follows by Definition 4 that
$(\mathcal{M}, w') \vDash \varphi$.
By Definition 3-3,
$(\mathcal{M}, w') \vDash \psi$.
By Definition 3-2, it follows that
$w' \in \|\psi\|$.
Therefore
$\|\varphi\| \subseteq \|\psi\|$.
By a similar method, we may take any world $w \in \|\psi\|$ and prove that $w \in \|\varphi\|$, and thereby prove that $\|\psi\| \subseteq \|\varphi\|$.
Therefore $\|\varphi\| = \|\psi\|$ and Lemma A1.1 is proved.

Lemma A1.2: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model. Then

$$\|\neg\varphi\| = W - \|\varphi\|.$$

Proof:
Let us take the truth set of a formula φ:

$$\begin{aligned} \|\neg\varphi\| &= \{w \in W : (\mathcal{M}, w) \vDash \neg\varphi\} && \text{by Definition 4} \\ &= \{w \in W : (\mathcal{M}, w) \nvDash \varphi\} && \text{by Definition 3-3} \end{aligned}$$

$$= W - \{w \in W : (\mathcal{M}, w) \vDash \varphi\}$$
$$= W - \|\varphi\| \qquad \text{by Definition 4.}$$

Thus Lemma A1.2 is proved.

Lemma A1.3 Let $\mathcal{M} = \langle W, N, V \rangle$ be a model. Then

$$\|\varphi \to \psi\| = \|\neg\varphi\| \cup \|\psi\|.$$

Proof:

$$\begin{aligned}
\|\varphi \to \psi\| &= \{w \in W{:}(\mathcal{M}, w) \vDash \varphi \to \psi\} && \text{By Definition 4.}\\
&= \{w \in W{:}(\mathcal{M}, w) \nvDash \varphi \text{ or } (\mathcal{M}, w) \vDash \psi\} && \text{By Definition 3-3.}\\
&= \{w \in W{:}(\mathcal{M}, w) \vDash \neg\varphi \text{ or } (\mathcal{M}, w) \vDash \psi\} && \text{By Definition 3-3.}\\
&= \{w \in W{:}(\mathcal{M}, w) \vDash \neg\varphi\} \cup \{w \in W{:}(\mathcal{M}, w) \vDash \psi\}\\
&= \|\neg\varphi\| \cup \|\psi\| \ .
\end{aligned}$$

Thus Lemma A1.3 is proved.

Lemma A1.4 Let $\mathcal{M} = \langle W, N, V \rangle$ be a model.

If $X \in N_Z(w,\mathrm{A})$ and $X \subseteq Y \subseteq W$ then $Y \in N_Z(w,\mathrm{A})$.

Proof:

Let us suppose that

$X \in N_Z(w,\mathrm{A})$,

$Y \in N_Z(w, \Phi)$, and

$X \subseteq Y \subseteq W$.

If we replace Φ with A in Definition 5b, we have:

If $X \in N_Z(w,\mathrm{A})$ and $(W–X) \cup Y \in N_Z(w,\mathrm{A})$, then $Y \in N_Z(w,\mathrm{A})$.

Since it is the case that $X \in N_Z(w,\mathrm{A})$ and by Definition 5g, $W \in N_Z(w,\mathrm{A})$, therefore it follows that

$Y \in N_Z(w,\mathrm{A})$.

Thus Lemma A1.4 is proved.

Lemma A1.5 Let $\mathcal{M} = \langle W, N, V \rangle$ be a model.

If $X \subseteq W$ and $X \notin N_Z(w, \Phi)$, then

$$\{w' \in W : X \notin N_Z(w', \Phi)\} \in N_B(w,\mathrm{A}).$$

Proof:
Let us suppose that $X \subseteq W$ and
$X \notin N_Z(w, \Phi)$.
By Definition 5d, there is a level Ψ such that $\Phi \neq \Psi$ and
$X \in N_Z(w, \Psi)$.
By Definition 5e, it follows that
$\{w' \in W : X \in N_Z(w', \Psi)\} \in N_B(w,\mathrm{A})$. .....(1)
Take any w' in $\{w' \in W : X \in N_Z(w', \Psi)\}$. Then $X \in N_Z(w', \Psi)$.
So by Definition 5c, $X \notin N_Z(w', \Phi)$ and therefore
$w' \in \{w' \in W : X \notin N_Z(w', \Phi)\}$.
Thus
$\{w' \in W : X \in N_Z(w', \Psi)\} \subseteq \{w' \in W : X \notin N_Z(w', \Phi)\}$.
So, by (1) and Lemma A1.4,
$\{w' \in W : X \notin N_Z(w', \Phi)\} \in N_B(w,\mathrm{A})$.
Thus Lemma A1.5 is proved.

Lemma A1.6 Let $\mathcal{M} = \langle W, N, V \rangle$ be a model. Then

$$\|\varphi \to \varphi\| = W.$$

Proof:

$$\begin{aligned} \|\varphi \to \varphi\| &= \{w \in W : (\mathcal{M}, w) \vDash \varphi \to \varphi\} && \text{By Definition 4} \\ &= \{w \in W : (\mathcal{M}, w) \nvDash \varphi \text{ or } (\mathcal{M}, w) \vDash \varphi\} && \text{By Definition 3-3} \\ &= W. \end{aligned}$$

Thus Lemma A1.6 is proved.

A.2. Axiom Lemmas

We shall now show that for any model and any world in that model, each of our axioms and properties are true.

$$\begin{aligned}&\mathrm{BA}\varphi \vee \mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \mathrm{BI}\varphi \vee \\ &\qquad \mathrm{BE}\varphi \vee \mathrm{BW}_n\varphi \vee \mathrm{BW}_{n-1}\varphi \vee \ldots \vee \mathrm{BW}_1\varphi \vee \mathrm{BD}\varphi.\end{aligned} \tag{A1}$$

Lemma A2.1: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, and Z∈{B,D,I}. Then

$(\mathcal{M}, w) \vDash \mathrm{ZA}\varphi \vee \mathrm{ZU}_1\varphi \vee \mathrm{ZU}_2\varphi \vee \ldots \vee \mathrm{ZU}_n\varphi \vee$
$\mathrm{ZI}\varphi \vee \mathrm{ZE}\varphi \vee \mathrm{ZW}_n\varphi \vee \mathrm{ZW}_{n-1}\varphi \vee \ldots \vee \mathrm{ZW}_1\varphi \vee \mathrm{ZD}\varphi.$

Proof:

Take any world $w \in W$, by Definition 4, there is a formula φ, such that,

$\|\varphi\| = \{w \in W : (\mathcal{M}, w) \vDash \varphi\}.$

By Definition 5d, it follows that for at least one level Φ where $\Phi \in \mathit{Levels}$,

$\|\varphi\| \in N_Z(w, \Phi),$

and therefore then,

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi.$

Therefore, by Definition 3-3, it follows that

$(\mathcal{M}, w) \vDash \mathrm{ZA}\varphi \vee \mathrm{ZU}_1\varphi \vee \mathrm{ZU}_2\varphi \vee \ldots \vee \mathrm{ZU}_n\varphi \vee$
$\mathrm{ZI}\varphi \vee \mathrm{ZE}\varphi \vee \mathrm{ZW}_n\varphi \vee \mathrm{ZW}_{n-1}\varphi \vee \ldots \vee \mathrm{ZW}_1\varphi \vee \mathrm{ZD}\varphi.$

Therefore, Lemma A.A1 is proved and axiom A1 verified.

$$\text{If } \Phi \neq \Psi, \text{then } \mathrm{B}\Phi\varphi \rightarrow \neg\mathrm{B}\Psi\varphi. \tag{A2}$$

Lemma A2.2: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, $\Phi, \Psi \in \mathit{Levels}$, and Z∈{B,D,I}.

If $\Phi \neq \Psi$ then $(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \rightarrow \neg\mathrm{Z}\Psi\varphi.$

Proof:

Take any world $w \in W$. Let us suppose that $\Phi \neq \Psi$ and that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi.$ ..... (1)

By Definition 3-2, it follows that

$\|\varphi\| \in N_Z(w, \Phi).$

Therefore by Definition 5c,

$\|\varphi\| \notin N_Z(w, \Psi).$

By Definition 3-2, it follows that

$(\mathcal{M}, w) \nvDash \mathrm{Z}\Psi\varphi$

and therefore by Definition 3-3 that

$(\mathcal{M}, w) \vDash \neg\mathrm{Z}\Psi\varphi.$ ..... (2)

From (1) and (2), it follows that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \rightarrow \neg\mathrm{Z}\Psi\varphi$.

Therefore, Lemma A.A2 is proved and so A2 is verified.

$$\text{If } \varphi \equiv \psi, \text{then } \mathrm{B}\Phi\varphi \equiv \mathrm{B}\Phi\psi. \tag{A3}$$

Lemma A2.3: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model, φ and ψ be formulae, $\Phi \in \mathit{Levels}$, and Z$\in\{$B,D,I$\}$.

If $(\mathcal{M}, w) \vDash \varphi \equiv \psi$ then $(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \equiv \mathrm{Z}\Phi\psi$.

Proof:

Take any world $w \in W$. Let us suppose

$(\mathcal{M}, w) \vDash \varphi \equiv \psi$, ... (1)

and that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi$. ..... (2)

By Definition 3-2, it follows from (2) that

$\|\varphi\| \in N_Z(w, \Phi)$. ..... (3)

By Lemma A1.1, it follows from (1) and (3) that

$\|\psi\| \in N_Z(w, \Phi)$.

Therefore, by Definition 3-2,

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\psi$. ..... (4)

Given (2) and (4), therefore it follows that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \rightarrow \mathrm{Z}\Phi\psi$.

By supposing (1) and (4), we can arrive at (2) by a similar manner. Therefore we can similarly prove that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\psi \rightarrow \mathrm{Z}\Phi\varphi$.

and therefore,

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \equiv \mathrm{Z}\Phi\psi$.

Thus Lemma A2.3 is proved and axiom A3 is verified.

$$\mathrm{BA}\varphi \equiv \mathrm{BD}\neg\varphi. \tag{A4}$$

$$\mathrm{BD}\varphi \equiv \mathrm{BA}\neg\varphi. \tag{P1}$$

$$\text{For each } i \in \{1, 2, \ldots, n\}, \mathrm{BU}_i\varphi \equiv \mathrm{BW}_i\neg\varphi. \tag{A5}$$

$$\text{For each } i \in \{1, 2, \ldots, n\}, \mathrm{BW}_i\varphi \equiv \mathrm{BU}_i\neg\varphi. \tag{P2}$$

$$\mathrm{BE}\varphi \equiv \mathrm{BE}\neg\varphi. \tag{A6}$$

$$\mathrm{BI}\varphi \equiv \mathrm{BI}\neg\varphi. \tag{A7}$$

Lemma A2.4: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, $\Phi \in \mathit{Levels}$, Φ^{M} be the mirror of Φ as defined in Definition 1.1, and Z$\in${B,D,I}, then

$$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \equiv \mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi.$$

Proof:

Take any world $w \in W$. Let us suppose that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi$. ..... (1)

By Definition 3-2, it follows that

$\|\varphi\| \in N_Z(w, \Phi)$.

By Definitions 5a and 1.1, it follows that

$W - \|\varphi\| \in N_Z(w, \Phi^{\mathrm{M}})$.

Therefore, by Lemma A1.2,

$\|\neg\varphi\| \in N_Z(w, \Phi^{\mathrm{M}})$.

By Definition 3, it then follows that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi$. ..... (2)

So, from (1) and (2), it follows that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \rightarrow \mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi$. ..... (3)

Let us suppose (2). By a similar method to that of how we derived (3) above, it follows that we can derive (1). Therefore, it follows that

$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi \rightarrow \mathrm{Z}\Phi\varphi$. ..... (4)

From (3) and (4), it follows that $(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \equiv \mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi$. Thus, Lemma A2.4 is proved.

Lemma A2.4 verifies that axioms A4, A5, A6, and A7 are correct in the semantics.

P1 and P2 follow from A4 and A5 respectively by replacing φ with $\neg\varphi$ and are thus similarly correct.

$$\mathrm{B}\Pi\varphi \rightarrow \neg\mathrm{BI}\varphi \wedge \neg\mathrm{BI}\neg\varphi \quad (\text{where } \Pi \in \mathit{Levels} - \{\mathrm{I}\}). \tag{P3}$$

Lemma A2.5: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model, φ be a formula, $\Pi \in \mathit{Levels} - \{\mathrm{I}\}$, and $\mathrm{Z} \in \{\mathrm{B,D,I}\}$, then

$$(\mathcal{M}, w) \vDash \mathrm{Z}\Pi\varphi \rightarrow \neg\mathrm{ZI}\varphi \wedge \neg\mathrm{ZI}\neg\varphi.$$

Proof:
Take any world $w \in W$. Let us suppose that
$(\mathcal{M}, w) \vDash \mathrm{Z}\Pi\varphi$.
By axiom A2, it follows that
$(\mathcal{M}, w) \vDash \neg\mathrm{ZI}\varphi$. ..... (1)
Therefore, it follows that
$(\mathcal{M}, w) \vDash \mathrm{Z}\Pi\varphi \rightarrow \neg\mathrm{ZI}\varphi$. ..... (2)
By axiom A7, it follows from (1) that $(\mathcal{M}, w) \vDash \neg\mathrm{ZI}\neg\varphi$, and thus
$(\mathcal{M}, w) \vDash \mathrm{Z}\Pi\varphi \rightarrow \neg\mathrm{ZI}\neg\varphi$. ..... (3)
It follows from (2) and (3) that
$(\mathcal{M}, w) \vDash \mathrm{Z}\Pi\varphi \rightarrow \neg\mathrm{ZI}\varphi \wedge \neg\mathrm{ZI}\neg\varphi$.
Therefore, Lemma A2.5 is proved and property P3 is verified.

$$\begin{aligned}\mathrm{BI}\varphi \rightarrow & \neg\mathrm{BA}\varphi \wedge \neg\mathrm{BU}_1\varphi \wedge \neg\mathrm{BU}_2\varphi \wedge \ldots \wedge \neg\mathrm{BU}_n\varphi \wedge \neg\mathrm{BE}\varphi \wedge \\ & \neg\mathrm{BE}\neg\varphi \wedge \neg\mathrm{BU}_n\neg\varphi \wedge \ldots \wedge \neg\mathrm{BU}_2\neg\varphi \wedge \neg\mathrm{BU}_1\neg\varphi \wedge \neg\mathrm{BA}\neg\varphi.\end{aligned} \tag{P4}$$

$$\begin{aligned}\mathrm{BI}\varphi \rightarrow & \neg\mathrm{BA}\varphi \wedge \neg\mathrm{BU}_1\varphi \wedge \neg\mathrm{BU}_2\varphi \wedge \ldots \wedge \neg\mathrm{BU}_n\varphi \wedge \neg\mathrm{BE}\varphi \wedge \\ & \neg\mathrm{BE}\neg\varphi \wedge \neg\mathrm{BW}_n\varphi \wedge \ldots \wedge \neg\mathrm{BW}_2\varphi \wedge \neg\mathrm{BW}_1\varphi \wedge \neg\mathrm{BD}\varphi.\end{aligned} \tag{P5}$$

Lemma A2.6: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model, φ be a formula, $\Pi \in \mathit{Levels} - \{\mathrm{I}\}$, and $\mathrm{Z} \in \{\mathrm{B,D,I}\}$, then
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi \rightarrow \neg\mathrm{Z}\Pi\varphi$.

Proof:
Take any world $w \in W$. Let us suppose that
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi$.
By Lemma A2.2, it follows that
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi \rightarrow \neg\mathrm{Z}\Pi\varphi$.
Thus Lemma A2.6 is proved.

If we expand $\mathrm{Z}\Pi\varphi$ to the concatenation of all the levels in Π, it follows that P5 is verified.
By using Lemma A2.4, P5 can be easily converted into P4, and so P4 is also verified.

$$\mathrm{BI}\varphi \equiv \neg\mathrm{PI}\varphi. \tag{P6}$$

Lemma A2.7: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, and Z$\in${B,D,I} (P from the belief axioms is represented here by $\mathrm{P_B}$), then
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi \equiv \neg\mathrm{P_Z I}\varphi$.

Proof:
Take any world $w \in W$. Let us suppose that
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi$.
By Lemma A2.4, $(\mathcal{M}, w) \vDash \neg\mathrm{ZI}\varphi$, and also the reverse as well, so therefore,
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi \equiv \neg\mathrm{ZI}\varphi$.
By Definition 1, it follows that
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi \equiv \neg\mathrm{P_Z I}\neg\neg\varphi$,
which is effectively equivalent to
$(\mathcal{M}, w) \vDash \mathrm{ZI}\varphi \equiv \neg\mathrm{P_Z I}\varphi$.
Thus Lemma A2.7 is proved and property P6 is verified.

$$\mathrm{P}\Phi\varphi \equiv \bigvee[\{\mathrm{B}\Psi\varphi : \Psi \in \mathit{Levels}\} - \{\mathrm{B}\Phi^{\mathrm{M}}\varphi\}]. \tag{P7}$$

Lemma A2.8: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, $\Phi \in \mathit{Levels}$, and Z$\in${B,D,I} (P from the belief axioms is represented here by $\mathrm{P_B}$), then
$(\mathcal{M}, w) \vDash \mathrm{P_Z I}\varphi \equiv \bigvee[\{\mathrm{Z}\Psi\varphi : \Psi \in \mathit{Levels}\} - \{\mathrm{Z}\Phi^{\mathrm{M}}\varphi\}]$.

Proof:
Let us take the case where $\Phi = \mathrm{A}$. Take any world $w \in W$. Let us suppose that
$(\mathcal{M}, w) \vDash \mathrm{P_B A}\varphi$. ..... (1)

By Definition 1,

$(\mathcal{M}, w) \vDash \neg\mathrm{BA}\neg\varphi$.

By property P1,

$(\mathcal{M}, w) \vDash \neg\mathrm{BD}\varphi$.

From axiom A2, we can infer that

$(\mathcal{M}, w) \vDash \mathrm{B\Pi}\varphi$ (where $\Pi \in \mathit{Levels} - \{\mathrm{D}\}$).

By Definition 3-3, it follows that this can also be presented as the disjunction of the Π belief levels, therefore

$$(\mathcal{M}, w) \vDash \mathrm{BA}\varphi \vee \mathrm{BU}_1\varphi \vee \mathrm{BU}_2\varphi \vee \ldots \vee \mathrm{BU}_n\varphi \vee \mathrm{BI}\varphi \vee \mathrm{BE}\varphi \vee \mathrm{BW}_n\varphi \vee \mathrm{BW}_{n-1}\varphi \vee \ldots \vee \mathrm{BW}_1\varphi. \quad \text{(2)}$$

So, it follows from (1) and (2) that

$$(\mathcal{M}, w) \vDash \mathrm{P_B I}\varphi \rightarrow \bigvee[\{\mathrm{B\Psi}\varphi : \Psi \in \mathit{Levels}\} - \{\mathrm{B\Phi^M}\varphi\}]. \quad \text{(3)}$$

If we supposed (2), we can derive (1) by a similar manner and therefore

$$(\mathcal{M}, w) \vDash \bigvee[\{\mathrm{B\Psi}\varphi : \Psi \in \mathit{Levels}\} - \{\mathrm{B\Phi^M}\varphi\}] \rightarrow \mathrm{P_B I}\varphi. \quad \text{(4)}$$

So, from (3) and (4),

$(\mathcal{M}, w) \vDash \mathrm{P_B I}\varphi \equiv \bigvee[\{\mathrm{B\Psi}\varphi : \Psi \in \mathit{Levels}\} - \{\mathrm{B\Phi^M}\varphi\}]$.

Thus Lemma A2.8 is proved for the case of $\Phi = \mathrm{A}$.

All for the other cases of Φ can be similarly proved without problem. Therefore, Lemma A2.8 is proved and P7 is verified.

$$\mathrm{B\Phi}(\varphi \rightarrow \psi) \rightarrow (\mathrm{BA}\varphi \rightarrow \mathrm{B\Phi}\psi). \quad \text{(A8)}$$

Lemma A2.9: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, $\Phi \in \mathit{Levels}$, and Z∈{B,D,I}, then

$(\mathcal{M}, w) \vDash \mathrm{Z\Phi}(\varphi \rightarrow \psi) \rightarrow (\mathrm{ZA}\varphi \rightarrow \mathrm{Z\Phi}\psi)$.

Proof:

Take any world $w \in W$. Let us suppose that

$$(\mathcal{M}, w) \vDash \mathrm{Z\Phi}(\varphi \rightarrow \psi). \quad \text{(1)}$$

We must show that $(\mathcal{M}, w) \vDash \mathrm{ZA}\varphi \rightarrow \mathrm{Z\Phi}\psi$. Let us also suppose that

$$(\mathcal{M}, w) \vDash \mathrm{ZA}\varphi. \quad \text{(2)}$$

Now we must show $(\mathcal{M}, w) \vDash \mathrm{Z\Phi}\psi$.

By Definition 3-2, it follows from (1) that

$$\|(\varphi \rightarrow \psi)\| \in N_Z(w, \Phi). \quad \text{(3)}$$

By Definition 3-2, it follows from (2) that

$\|\varphi\| \in N_Z(w,\mathrm{A})$. ..... (4)

By Lemma A1.3, it follows from (3) that

$\|\neg\varphi\| \cup \|\psi\| \in N_Z(w,\Phi)$. ..... (5)

By Lemma A1.2, it follows from (5) that,

$(W-\|\varphi\|) \cup \|\psi\| \in N_Z(w,\Phi)$. ..... (6)

By Definition 5b, it follows from (4) and (6) that,

$\|\psi\| \in N_Z(w,\Phi)$.

Therefore, by Definition 3-2,

$(\mathcal{M},w) \vDash \mathrm{Z}\Phi\psi$.

So, $\mathcal{M},w) \vDash \mathrm{Z}\Phi(\varphi \to \psi) \to (\mathrm{ZA}\varphi \to \mathrm{Z}\Phi\psi)$.

Thus, Lemma A2.9 is proved and axiom A8 is verified.

$$\mathrm{B}\Pi\varphi \to \mathrm{P}\Pi\varphi \quad \text{where } \Pi \in \mathit{Levels}-\{\mathrm{E,I}\}. \tag{P8}$$

Lemma A2.10: Let $\mathcal{M} = \langle W,N,V\rangle$ be a model, φ be a formula, $\Pi \in \mathit{Levels}-\{\mathrm{E,I}\}$, and Z$\in\{B,D,I\}$ (P from the belief axioms is represented here by $\mathrm{P_B}$), then

$(\mathcal{M},w) \vDash \mathrm{Z}\Pi\varphi \to \mathrm{P_Z}\Pi\varphi$.

Proof:

Take any world $w \in W$. Let us suppose that

$(\mathcal{M},w) \vDash \mathrm{Z}\Pi\varphi$.

By Lemma A2.2 and Definition 1.1, it follows that,

$(\mathcal{M},w) \vDash \neg\mathrm{Z}\Pi^{\mathrm{M}}\varphi$.

By Lemma A2.4, it follows that

$(\mathcal{M},w) \vDash \neg\mathrm{Z}\Pi\neg\varphi$.

By Definition 1, it follows that

$(\mathcal{M},w) \vDash \neg\neg\mathrm{P_Z}\Pi\neg\neg\varphi$,

which is equivalent to

$(\mathcal{M},w) \vDash \mathrm{P_Z}\Pi\varphi$.

Therefore

$(\mathcal{M},w) \vDash \mathrm{Z}\Pi\varphi \to \mathrm{P_Z}\Pi\varphi$.

Thus, Lemma A2.10 is proved and P8 is verified.

$$\mathrm{B}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{B}\Phi\varphi. \tag{A9}$$

Lemma A2.11: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model, φ be a formula, $\Phi \in \mathit{Levels}$, and Z∈{B,D,I}, then

$$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{Z}\Phi\varphi.$$

Proof:
Take any world $w \in W$. Let us suppose that
$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi$. ..... (1)
By Definition 3.2, it follows that
$\|\varphi\| \in N_Z(w, \Phi)$.
By Definition 5e, it then follows that
$\{w' : \|\varphi\| \in N_Z(w', \Phi)\} \in N_B(w,\mathrm{A})$.
So, by Definition 3-2,
$\{w' : (\mathcal{M}, w') \vDash \mathrm{Z}\Phi\varphi\} \in N_B(w,\mathrm{A})$.
Again, by Definition 3-2, it follows that
$\|\mathrm{Z}\Phi\varphi\| \in N_B(w, \Phi)$,
and therefore (again by Definition 3-2),
$(\mathcal{M}, w) \vDash \mathrm{BA}\ \mathrm{Z}\Phi\varphi$. ..... (2)
So it follows from (1) and (2) that
$(\mathcal{M}, w) \vDash \mathrm{Z}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{Z}\Phi\varphi$.
Thus Lemma A2.11 is proved and A9 is verified.

$$\mathrm{P}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{P}\Phi\varphi. \tag{A10}$$

Lemma A2.12: Let $\mathcal{M} = \langle W, N, V \rangle$ be a model, φ be a formula, $\Phi \in \mathit{Levels}$, and Z∈{B,D,I} (P from the belief axioms is represented here by $\mathrm{P_B}$), then

$$(\mathcal{M}, w) \vDash \mathrm{P_Z}\Phi\varphi \rightarrow \mathrm{BA}\ \mathrm{P_Z}\Phi\varphi.$$

Proof:
Take any world $w \in W$. Let us suppose that
$(\mathcal{M}, w) \vDash \mathrm{P_Z}\Phi\varphi$. ..... (1)

By Definition 1, it follows that

$(\mathcal{M}, w) \vDash \neg \mathrm{Z}\Phi\neg\varphi$.

By Lemma A2.4 it follows that

$(\mathcal{M}, w) \vDash \neg \mathrm{Z}\Phi^{\mathrm{M}}\varphi$.

By Definition 3-2, therefore

$\|\varphi\| \notin N_Z(w, \Phi^{\mathrm{M}})$.

By Definition 5f,

$\{w' : \|\varphi\| \notin N_Z(w', \Phi^{\mathrm{M}})\} \in N_B(w,\mathrm{A})$.

By Definition 3-2,

$\{w' : (\mathcal{M}, w) \nvDash \mathrm{Z}\Phi^{\mathrm{M}}\varphi\} \in N_B(w,\mathrm{A})$.

By Definition 3-3,

$\{w' : (\mathcal{M}, w) \vDash \neg\mathrm{Z}\Phi^{\mathrm{M}}\varphi\} \in N_B(w,\mathrm{A})$.

By Definition 1 and Lemma A2.4, it then follows that

$\{w' : (\mathcal{M}, w) \vDash \mathrm{P_Z}\Phi\varphi\} \in N_B(w,\mathrm{A})$.

So, by Definition 3-2,

$(\mathcal{M}, w) \vDash \mathrm{BA}\ \mathrm{P_Z}\Phi\varphi$. ..... (2)

Therefore, from (1) and (2) it follows that

$(\mathcal{M}, w) \vDash \mathrm{P_Z}\Phi\varphi \to \mathrm{BA}\ \mathrm{P_Z}\Phi\varphi$.

Thus, Lemma A2.12 is proved and A10 is verified.

$$\mathrm{BA}(\varphi \to \varphi). \tag{A11}$$

Lemma A2.13: Let $\mathcal{M} = \langle W, N, V\rangle$ be a model, φ be a formula, $w \in W$, and Z∈{B,D,I}, then

$$(\mathcal{M}, w) \vDash \mathrm{ZA}(\varphi \to \varphi).$$

Proof:

Take any world $w \in W$. Let us examine

$(\mathcal{M}, w) \vDash \mathrm{ZA}(\varphi \to \varphi)$. ..... (1)

By Definition 3-2, (1) is true if and only if

$\|\varphi \to \varphi\| \in N_Z(w,\mathrm{A})$. ..... (2)

By Lemma A1.6, (2) is true if and only if

$W \in N_Z(w,\mathrm{A})$. ..... (3)

But, by Definition 5g, (3) is true, therefore (1) is also true.

Therefore, Lemma A2.13 is proved, and A11 is verified.

B. Completeness Proofs

The proof that the logic characterised by Axioms A1–A11 is complete by the class of minimal models satisfying Definition 5 is a standard proof based on canonical models (see [11] for the basic structure of the proof and standard results). What we are going to show here is that the conditions given in Definition 5 corresponds to the axioms.

Definition 1. *Let* $\mathcal{M} = \langle W, \{N_B, N_D, N_I\}, V\rangle$ *be a minimal model.* $\mathcal{M}$ *is a* canonical *model iff:*

1. *W is the set of all maximal consistent sets of formulae;*
2. *For $Z \in \{B, D, I\}$*

$$N_Z(w, \Phi) = \begin{cases} \{|\varphi| : \mathrm{Z}\Phi\varphi \in w\} & \Phi \neq \mathrm{I} \\ \{|\varphi| : \mathrm{Z}\Phi\varphi \in w\} \cup \{X : \forall\varphi(X \neq |\varphi|)\} & \Phi = \mathrm{I} \end{cases}$$

   where $|\varphi| = \{w \in W : \varphi \in w\}$
3. $V(p, w) = true$ *iff* $p \in w$, *for* $p \in \mathcal{P}$.

Contrary to what happens for Kripke models we can have multiple minimal canonical models. They can vary from the smallest minimal model where we have that each neighbourhood is restricted to the case given for $\Phi \neq \mathrm{I}$, to the case where each neighbourhood is defined as in the case for $\Phi = \mathrm{I}$ (i.e., when we have the largest minimal model). Notice that in Definition 1 the choice of I is arbitrary, all we need is to capture all possible subsets of X and to ensure that they are distributed over the neighbourhoods for the various level of modalities.

All we have to do to prove completeness is to show that each time an axiom belongs to a possible world (and an axiom as such belongs to all possible world), then the corresponding semantic condition is satisfied by the canonical model.

Axiom A1: The condition corresponding to this axiom is condition d) of Definition 5.

Suppose that condition d) does not hold. This means that there is a set $X \subseteq W$ such that for every $\Phi \in Levels$, $X \notin N_Z(w, \Phi)$ for some world w. We have two cases, (i) for all formulas φ, $X \neq |\varphi|$ or (ii) $X = |\varphi|$

for some formula φ. For (i) according to the construction of the canonical model, $X \in N_Z(w, \mathrm{I})$. Contradiction. For (ii), since $X \notin N_Z(w, \Phi)$ for all $\Phi \in Levels$, then $|\varphi| \notin N_Z(w, \Phi)$, thus $\mathrm{Z}\Phi\varphi \notin w$. By maximality $\neg\mathrm{Z}\Phi\varphi \in w$, for all $\Phi \in Levels$, thus $\bigwedge_{\Phi \in Levels} \neg\mathrm{Z}\Phi\varphi \in w$. But $\bigwedge_{\Phi \in Levels} \neg\mathrm{Z}\Phi\varphi \in w \equiv \neg \bigvee_{\Phi \in Level} \mathrm{Z}\Phi\varphi$, and this implies that $\neg\mathsf{A1}, \mathsf{A1} \in w$ contrary to the consistency of w. Contradiction.

Axiom A2: The condition corresponding to this axiom is condition c) of Definition 5.

Suppose that the condition does not hold. This means that there is a set of possible worlds $X \in N_Z(w, \Phi) \cap N_Z(w, \Psi)$ for $\Phi \neq \Psi$. Since sets of worlds not corresponding to any proof set are only $N_Z(w, \mathrm{I})$, we have that $X = |\varphi|$ for some formula φ. But this means the $|\varphi| \in N_Z(w, \Phi)$ and $|\varphi| \in N_Z(w, \Psi)$, which means $\mathrm{Z}\Phi\varphi \in w$ and $\mathrm{Z}\Psi\varphi \in w$, from which we get that $\perp \in w$, contrary to the consistency of w. Contradiction. Thus $N_Z(w, \Phi) \cap N_Z(w, \Psi) = \{\}$.

Axiom A3: This axiom corresponds to the inference rule RE which is valid in every minimal model.

Axioms A4, A5, A6 and A7: Axioms A4, A5, A6 and A7, are all instances of the more general axiom

$$\mathrm{Z}\Phi\varphi \equiv \mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi. \tag{A4-7}$$

The condition that corresponds to this axiom is condition a) of Definition 5.

Suppose that we have a set of worlds X in $N_Z(w, \Phi)$. For $\Phi \neq \mathrm{I}$, by construction, $N_Z(\Phi, w) = \{|\varphi| : \mathrm{Z}\Phi\varphi \in w\}$, thus $X = |\varphi|$ for some formula φ, such that $\mathrm{Z}\Phi\varphi \in w$. From A4-7 we obtain $\mathrm{Z}\Phi^{\mathrm{M}}\neg\varphi \in w$, and then, by construction of the canonical model, $|\neg\varphi| \in N_Z(w, \Phi^{\mathrm{M}})$, which is $W - |\varphi| \in N_Z(w, \Phi^{\mathrm{M}})$. For $\Phi = \mathrm{I}$ we have to consider the case where X does not correspond to any proof set. But this means that also $W - X$ does not correspond to any proof set, otherwise X would correspond to a proof set. Therefore according to the construction of the canonical model both X and $W - X$ are in $N_Z(w, \mathrm{I})$. However, $\mathrm{I}^{\mathrm{M}} = \mathrm{I}$. Thus the property holds also in this case.

Axiom A8: The condition corresponding to this axiom is condition b) of Definition 5.

Suppose $\mathrm{ZA}\varphi \wedge \mathrm{Z}\Phi(\varphi \to \psi) \in w$. Accordingly, we have $\mathrm{ZA}\varphi \in w$, $\mathrm{Z}\Phi(\varphi \to \psi) \in w$ and by the axiom A8 and the maximality of w, $\mathrm{Z}\Phi\psi \in w$. From the first two we have $|\varphi| \in N_Z(w, \mathrm{A})$, and $|\varphi \to \psi| \in N_Z(w, \Phi)$. The latter is equivalent to $(W - |\varphi|) \cup |\psi| \in N_Z(w, \Phi)$, while $\mathrm{Z}\Phi\psi \in w$ means $|\psi| \in N_Z(w, \Phi)$. Thus condition b) of Definition 5 is satisfied.

Axiom A9: The conditions corresponding to this axiom are conditions e) and g) of Definition 5.

If $X \in N_Z(w, \Phi)$, for $\Phi \neq \mathrm{I}$, then we have that $X = |\varphi|$ for some formula φ. But this means that $\mathrm{Z}\Phi\varphi \in w$, and so BA $\mathrm{Z}\Phi\varphi \in w$ which, by the construction of the canonical model corresponds to what we want to prove. For I we have to consider the case that X is not the proof set for any formula φ. However, by construction of the canonical model, $\{w : X \in N_Z(w, \mathrm{I})\} = W$, and $W \in N_B(w, \mathrm{A})$.

Axiom A10: The conditions corresponding to this axiom are conditions f) and g) of Definition 5.

If $X \notin N_Z(w, \Phi)$, then for $\Phi = \mathrm{I}$, this means that $X = |\varphi|$ for some formula φ. Therefore $\neg\mathrm{ZI}\varphi \in w$, and so BA $\neg\mathrm{ZI}\varphi \in w$, so $|\neg\mathrm{ZI}\varphi| \in N_B(w, \mathrm{A})$ which is $\{w' : X \notin N_Z(w', \mathrm{I})\} \in N_B(w, \mathrm{A})$.

For $\Phi \neq \mathrm{I}$, in addition to the case above, we have to consider the case where X is not a proof set. But then, by construction, $\{w : X \notin N_Z(w, \Phi)\} = W$, and then we have that $W \in N_B(w, \mathrm{A})$.

Axiom A11: The condition corresponding to this axiom is condition g) of Definition 5.

$\mathrm{ZA}(\varphi \to \varphi) \in w$ means that $|\varphi \to \varphi| \in N_Z(w, \mathrm{A})$, and $|\varphi \to \varphi| = W$, thus $W \in N_Z(w, \mathrm{A})$.

C. Agent Verification

We shall now verify that the System defined in Definitions 6 to 8 is consistent with our Model as defined in Definitions 2 and 5. Let $\mathcal{M} = \langle W, N, V \rangle$ be a model, $\mathcal{A} = \langle \beta, \delta, \iota \rangle$ be an Agent, $\mathcal{S} = \langle \mathcal{M}, \mathcal{A} \rangle$ be a System, $\Phi, \Psi \in Levels$, $\gamma \in \{\beta, \delta, \iota\}$, $Z \in \{B, D, I\}$, and $w \in W$.

Definition 7 states that N_Z is a set made of subsets of W. This is consistent with the original signature of N_Z in Definition 2-2.

We will now examine the detailed construction of N_Z in the System and Agent in relation to the conditions placed upon N in Definition 5. We recall that $N_Z(w, \Phi)$ is as defined in Definition 8 and γ is as defined in Definition 7. Given that Z $\in$\{B,D,I\}, let $Z' = \gamma$.
For some set $X \subseteq W$ and $\Phi \in Levels$, let us suppose that

$X \in N_Z(w, \Phi)$

for some $\varphi \in \mathcal{L}$ such that $X = \|\varphi\|$.
To show consistency with Definition 5a, we need to prove

$W\text{–}X \in N_Z(w, \Phi^{\mathrm{M}})$. ..... (1)

Let us examine the case where $\Phi = \mathrm{A}$. By Definition 8a,

$Z'(w, X) = \gamma(w, X) = 1$.

By Definition 7b, $\gamma(w, X) = 1 - \gamma(w, W\text{–}X)$, therefore

$Z'(w, W\text{–}X) = 1 - Z'(w, X) = 0$.

By Definition 8e then,

$W\text{–}X \in N_Z(w, \mathrm{D})$.

By Definition 1.1, D is the mirror of A, so (1) holds for $\Phi = \mathrm{A}$.
Let us examine the case of $\Phi = \mathrm{U}_i$. By Definition 8b, $Z'(w, X) \in \mathbb{Q}(\mathrm{U}_i)$, so

$Z'(w, X) = m$,

where $\mathbb{Q}(\mathrm{U}_i)$ is as defined in Definition 6-2, $m \in \mathbb{Q}$, and $0{\cdot}5 < m < 1$.
By Definition 7b,

$Z'(x, W\text{–}X) = 1 - Z'(w, X) = 1 - m$.

This aligns with $\mathbb{Q}(\mathrm{W}_i)$ in Definition 6-2, so by Definition 8d,

$W\text{–}X \in N_Z(w, \mathrm{W}_i)$.

By Definition 1.1, W_i is the mirror of U_i, so (1) holds for $\Phi = \mathrm{U}_i$. All the other cases can be similarly verified, so therefore, we can verify that Definitions 6 to 8 are consistent with, and 7b is necessary for, Definition 5a.

We will now examine consistency with Definition 5b. After reminding

ourselves that $Z' = \gamma$, let us suppose that

$X \in N_Z(w,\mathrm{A})$ and $((W–X) \cup Y) \in N_Z(w, \Phi)$

for some formulas $\varphi, \psi \in \mathcal{L}$ such that $\|\varphi\| = X$ and $\|\psi\| = Y$.
To show consistency with 5b, we must show that

$Y \in N_Z(w, \Phi)$.

By Definition 8a

$\gamma(w, X) = Z'(w, X) = 1$.

By Definition 7c,

$\gamma(w, Y) = Z'(w, Y) = \gamma(w, (W–X) \cup Y)$.

Since $((W–X) \cup Y) \in N_Z(w, \Phi)$, by Definition 8, it follows that

$Y \in N_Z(w, \Phi)$.

Thus, we can verify that Definitions 6 to 8 are consistent with, and Definition 7c is necessary for, Definition 5b.

We will now examine consistency with Definition 5c and 5d. By Definition 8, the possible value of $Z'(w, X)$ is in the range defined by Definition 6. It can be easily seen by reading Definition 6 that no γ/Z' value assigned to one level could possibly be assigned to another level. It can be similarly seen that each γ/Z' value must be assigned to a level and cannot be unassigned. Therefore, we can verify that Definitions 7 and 8 are consistent with Definition 5, subparts c and d.

We will now examine the System's consistency with Definition 5e. Let us suppose $Z' = \gamma$ and that $X \in N_Z(w, \Phi)$. We need to therefore show that

$\{w' \in W : X \in N_Z(w', \Phi)\} \in N_B(w,\mathrm{A})$

for some formula $\varphi \in \mathcal{L}$ such that $\|\varphi\| = X$.
Let $Z'(w, X) = q$, where $q \in \{x \in \mathbb{Q} : 0 \leq x \leq 1\} \cup \{\boldsymbol{u}\}$.
Then, by Definition 7d,

$\beta(w, \{w' \in W : Z'(w', X) \approx q\}) = 1$.

By Definitions 6-3 and 8, $Z'(w', X) \approx q$ iff $X \in N_Z(w, \Phi)$. Therefore

$\beta(w, \{w' \in W : X \in N_Z(w', \Phi)\}) = 1$.

By Definition 8a,

$\{w' \in W : X \in N_Z(w', \Phi)\} \in N_B(w,\mathrm{A})$.

Therefore, we can verify that Definitions 6 to 8 are consistent with, and Definition 7d is necessary for, Definition 5e.

Lemma A1.5 proves that Definition 5f is derivable from 5e, therefore, we can verify that Definitions 6 to 8 are also consistent with Definition 5f.

$\gamma(w, W) = 1$ from Definition 7c and Definition 8a easily show that the Agent System is consistent with Definition 5g.

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