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T violation and the unidirectionality of time

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Abstract An increasing number of experiments at the Belle, BNL, CERN, DAΦNE and SLAC accelerators are confirming the violation of time reversal invariance (T). The violation signifies a fundamental asymmetry between the past and future and calls for a major shift in the way we think about time. Here we show that processes which violate T symmetry induce destructive interference between different paths that the universe can take through time. The interference eliminates all paths except for two that represent continuously forwards and continuously backwards time evolution. Evidence from the accelerator experiments indicates which path the universe is effectively following. This work may provide fresh insight into the long-standing problem of modeling the dynamics of T violation processes. It suggests that T violation has previously unknown, large-scale physical effects and that these effects underlie the origin of the unidirectionality of time. It may have implications for the Wheeler-DeWitt equation of canonical quantum gravity. Finally it provides a view of the quantum nature of time itself.

Keywords CP violation · T violation · Kaons · Arrow of time · Quantum interference · Quantum foundations · Wheeler-DeWitt equation

1 Introduction

The physical nature of time has been an enigma for centuries. The main tools for discussing its unidirectionality are the phenomenological arrows of time. A great deal of progress has been made in recent decades in linking various arrows together [1]. A notable exception is the matter-antimatter arrow which arises from the violation of charge-parity conjugation invariance (CP) in meson decay [2,3,4]. CP violation, which was first discovered by Cronin, Fitch and coworkers [5] in 1964 in the decay of neutral kaons (K mesons), provides a clue to the origin of the large-scale matter-antimatter

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imbalance of the universe [6]. The combined CPT invariance has been confirmed in kaon decay to a high degree of precision [7] suggesting that neutral kaon decay also violates T invariance. To date no violation of CPT has been observed in B meson decay [8] and so the CP violation in B meson decay also appears to be consistent with T violation. But more importantly, experimental evidence has confirmed the direct T violation in K meson decay [9] independently of CPT invariance. Nonetheless, T violating processes are relatively rare and the magnitudes of the violations are relatively small, and so despite their important asymmetric temporal nature, they are often regarded as having little impact on the nature of time. As a result, the physical significance of T violation has remained obscure.

The CP violation due to the weak interaction is described in the Standard Model of particle physics by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [10,11]. Nevertheless, a fundamental and intriguing question remains open. In principle, the CKM model provides the basis for the time evolution of mesons in terms of Schrödinger's equation and an associated Hamiltonian H. The evidence of T violation implies that $THT^{-1} \neq H$ where T represents the time reversal operation [12]. But how should one incorporate the *two* Hamiltonians, H for time evolution towards our future and THT^{-1} for evolution towards our past, in one quantum equation of motion? At issue is the fact that Schrödinger's equation describes consistent unitary evolution in both directions of time: the unitary evolution by negative time intervals merely "back tracks" exactly the same time evolution described by positive time intervals. Yet, paradoxically, the evidence of T violation processes appears to suggests that Nature does not work in this way in general.

Currently, to model T violating processes we must choose the direction of time evolution and a specific version, say H, of the Hamiltonian in order to write down the Schrödinger equation. By applying the time reversal operation we then obtain the Schrödinger equation involving the THT^{-1} version of the Hamiltonian for evolution in the opposite direction of time. But at any point we are restricted to having a dynamical equation of motion for a specific direction of time evolution and a corresponding specific version of the Hamiltonian. We do not have a dynamical equation of motion for the situation where the *direction of time evolution cannot be specified*. In this case there is no argument for favoring one version of the Hamiltonian over the other and so if one version of the Hamiltonian is to appear in the equation of motion then so must the other. This problem becomes critical when we attempt to describe the universe as a closed system as needed in cosmology, for then the direction of time evolution cannot be specified because a closed system precludes any external clock-like device for use as a reference for the direction of time. There is, therefore, no satisfactory quantum formalism for describing a universe that exhibits T violation processes. This failure of standard quantum theory is all the more serious given that CP and, by implication, T violation play a critical role in baryogenesis in the early history of the universe [6,13]. It is clearly an important problem that lies at the very foundations of quantum theory. Its resolution calls for a major shift in the way we think about time and dynamical equations of motion.

We address this issue by deriving a single quantum dynamical equation which incorporates the two versions of a T-violating Hamiltonian. We begin by using Feynman's sum over histories [14] method to construct the set of all possible paths that the uni-

verse can take through time. The set represents every possible evolution zigzagging forwards and backwards through time. We then show how T violation induces destructive interference that eliminates most of the possible paths that the universe can follow. Effectively only two main paths, representing continuously forwards and continuously backwards time evolution, survive the interference. Physical evidence from accelerator experiments allows us to to *distinguish between the two directions of time* and determine which path the universe is effectively following. In short, T violation is shown to have *a previously unknown large-scale physical effect that underlies the unidirectionality of time*. This is the main result of this work. We end with a discussion that includes the potential impact for cosmology and the Wheeler-DeWitt equation in canonical quantum gravity.

2 Possible paths through time

Consider a system, which we shall refer to simply as the universe, whose composition in terms of size, matter and fields is consistent with the visible portion of the physical universe. For clarity we assume that the universe is closed in the sense that it does not interact with any other physical system and further, that there is no physical system external to the universe. This means that there is no clock external to the universe and so our analysis needs to be unbiased with respect to the direction of time evolution. Nevertheless it is convenient to differentiate two directions of time as "forward" and "backward" associated with evolution in the positive$-t$ and negative$-t$ directions of the time axis, respectively, although we stress that neither direction necessarily has any connection to our actual everyday experience. Also, we begin our analysis with the universe in a state $|\psi_0\rangle$ which we shall call the "origin state" without reference to the direction of time. With this in mind we express the evolution of the universe in the forwards direction over the time interval τ as $|\psi_F(\tau)\rangle = U_F(\tau)|\psi_0\rangle$ where

$$U_F(\tau) = \exp(-i\tau H_F) \tag{1}$$

is the forwards time evolution operator, and H_F is the Hamiltonian for forwards time evolution. Throughout this article we use units in which $\hbar = 1$. If time reversal symmetry were obeyed, the evolution in the opposite direction of time would be given by $\exp(i\tau H_F)$. To accommodate the time asymmetry of T violation processes, we need to replace H_F in this expression with its time reversed form. Thus we define the state of the universe after backwards evolution over a time interval of the same magnitude by $|\psi_B(\tau)\rangle = U_B(\tau)|\psi_0\rangle$ where

$$U_B(\tau) = \exp(i\tau H_B)\ , \tag{2}$$

and $H_B = TH_FT^{-1}$ is the Hamiltonian for backwards time evolution. This gives $U_B(\tau) = TU_F(\tau)T^{-1}$, in accord with Wigner's definition of time reversal [12]. In other words, H_F and H_B are the generators of time translations in the positive$-t$ and negative$-t$ directions, respectively.

The parameter τ in these expressions represents a time interval that could be observed using clock devices which are internal to the universe. A simple model of a clock for our purposes is a device which evolves unitarily through a sequence of orthogonal states at sufficient regularity to give a measure of a duration of time to any desired

accuracy. The orthogonal states would represent the eigenstates of a pointer observable which acts a reference for time. A harmonic oscillator is a device of this kind [15]. In order to avoid ambiguities in the definition of time intervals for different time directions, we assume that the Hamiltonian $H^{(\mathrm{clk})}$ describing our clock model is T invariant and so $T\,H^{(\mathrm{clk})}\,T^{-1} = H^{(\mathrm{clk})}$ during its normal operation. This gives an unambiguous operational meaning of the parameter τ as a time interval. We shall refer to the readings of such clocks as "clock time".

The matrix elements $\langle\phi|U_F(\tau)|\psi_0\rangle$ and $\langle\phi|U_B(\tau)|\psi_0\rangle$ represent the probability amplitudes for the universe in state $|\psi_0\rangle$ to evolve over the time interval τ to $|\phi\rangle$ via two paths in time corresponding to the forwards and backwards directions, respectively. Given that we have no basis for favoring one path over the other, we follow Feynman [14] and attribute an equal statistical weighting to each. Thus, *the total probability amplitude for the universe to evolve from one given state to another is proportional to the sum of the probability amplitudes for all possible paths through time between the two states.* In the current situation we have only two possible paths and so the total amplitude is proportional to

$$\langle\phi|U_F(\tau)|\psi_0\rangle + \langle\phi|U_B(\tau)|\psi_0\rangle = \langle\phi|U_F(\tau) + U_B(\tau)|\psi_0\rangle \ . \tag{3}$$

This result holds for all states $|\phi\rangle$ of the universe and so the unbiased time evolution of $|\psi_0\rangle$ over the time interval τ can be written as

$$|\Psi(\tau)\rangle = [U_F(\tau) + U_B(\tau)]\,|\psi_0\rangle \tag{4}$$

which we call the *symmetric time evolution* of the universe. To reduce unnecessary detail we use un-normalized vectors to represent states of the universe. We could have included in (3) an additional phase factor $e^{i\theta}$ to represent an arbitrary relative phase between the two paths, such as $\langle\phi|U_F(\tau)|\psi_0\rangle + e^{i\theta}\langle\phi|U_B(\tau)|\psi_0\rangle$, but the factor has no net physical effect for our analysis and so we omit it.

It follows that the symmetric time evolution of the universe in state $|\Psi(\tau)\rangle$ over an additional time interval of τ is given by

$$\begin{aligned}|\Psi(2\tau)\rangle &= [U_F(\tau) + U_B(\tau)]\,|\Psi(\tau)\rangle \\ &= [U_F(\tau) + U_B(\tau)]^2\,|\psi_0\rangle \ .\end{aligned}$$

Repeating this for N such time intervals yields

$$|\Psi(N\tau)\rangle = [U_F(\tau) + U_B(\tau)]^N\,|\psi_0\rangle \ . \tag{5}$$

Figure 1 gives a graphical interpretation of this result in terms of a binary tree. It is useful to write the expansion of the product on the right side as

$$|\Psi(N\tau)\rangle = \sum_{n=0}^{N} S_{N-n,n}\,|\psi_0\rangle \tag{6}$$

where $S_{m,n}$ represents a sum containing $\binom{n+m}{n}$ different terms each comprising n factors of $U_F(\tau)$ and m factors of $U_B(\tau)$, and where $\binom{k}{j} = k!/[(k-j)!j!]$ is the binomial coefficient. $S_{m,n}$ is defined by the recursive relation

$$S_{m,n} = \sum_{k=0}^{m} S_{m-k,n-1} U_F(\tau) U_B(k\tau) \tag{7}$$

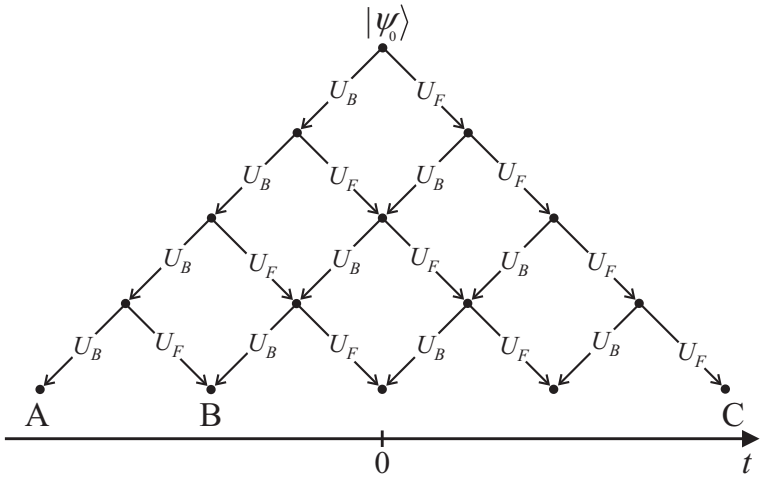

Fig. 1 Binary tree representation of the generation of the state $|\Psi(N\tau)\rangle$ from the origin state $|\psi_0\rangle$ according to (5) with $N = 4$. States are represented as nodes (solid discs) and unitary evolution by links (arrows) between them. The root node (at the top) represents the origin state $|\psi_0\rangle$ and the leaf nodes (on the bottom row) represent components of the state $|\Psi(4\tau)\rangle$ where, for example, the node labeled A represents the state $S_{4,0}|\psi_0\rangle = U_B^4|\psi_0\rangle$, B represents the state $S_{3,1}|\psi_0\rangle = [U_B^3U_F + U_B^2U_FU_B + U_BU_FU_B^2 + U_FU_B^3]|\psi_0\rangle$ and C represents $S_{0,4}|\psi_0\rangle = U_F^4|\psi_0\rangle$. The time axis labeled t at the bottom of the figure represents clock time which increases from left to right. Clock time is measured by clock devices whose Hamiltonian $H^{(\mathrm{clk})}$ is T invariant.

with $S_{m,0} = U_B(m\tau)$ and gives, for example, $S_{m,1} = \sum_{k=0}^{m} U_B[(m-k)\tau]U_F(\tau)U_B(k\tau)$.

The expression $\langle\phi|S_{N-n,n}|\psi_0\rangle$ represents the evolution of the universe from $|\psi_0\rangle$ to $|\phi\rangle$ over a set of $\binom{N}{n}$ paths through time, where each path comprises a total of n steps in the forwards direction and $N-n$ steps in the backwards direction. The set of paths includes all possible orderings of the forwards and backwards steps. This set of paths is the focus of the remaining analysis.

First consider the size of the time steps τ. To explore the consequences of the limit $\tau \to 0$ of infinitely-small time steps for a fixed total time t_{tot}, we set $\tau = t_{\mathrm{tot}}/N$ and consider (5) for increasing values of N. We can write $2^{-N}[U_F(\tau) + U_B(\tau)]^N = \{\exp[-i\frac{1}{2}(H_F - H_B)\tau] + \mathcal{O}(\tau^2)\}^N$ which becomes $\exp[-i\frac{1}{2}(H_F - H_B)t_{\mathrm{tot}}]$ as $N \to \infty$. In this limit, the universe evolves according to the Hamiltonian $\frac{1}{2}(H_F - H_B)$. This means that isolated subsystems of the universe that obey T invariance, such as our model of a clock, would not exhibit any evolution. As clocks are taken to measure time intervals, the universe as a whole would not exhibit dynamics in the conventional sense in this $\tau \to 0$ limit. The lack of dynamics is an unphysical result if our system is to model the visible universe. This suggests that the time interval τ should be a small non-zero number for a universe-like system. Accordingly, in the following we set the value of τ to be the smallest physically-reasonable time interval, the Planck time, i.e. $\tau \approx 5 \times 10^{-44} s$.

We now derive a more manageable expression for $S_{m,n}$. The first step is to reorder the expression so that all the U_B factors are to the left of the U_F factors using the Zassenhaus formula [26]. This yields

$$S_{m,n} = U_B(m\tau)U_F(n\tau)\sum_{v=0}^{m}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell}\exp\Big[(v+\cdots+\ell+k)\tau^2[H_F,H_B]\Big]\exp[\mathcal{O}(\tau^3)] \quad (8)$$

where there are n summations on the right side and $[A,B]$ is the commutator of A and B. Appendix A gives the details of the calculation. Next, the analysis is made simpler

by expressing the summand in the eigenbasis of the Hermitian operator $i[H_F, H_B]$. For this we use the resolution of the identity given by $\mathbf{1} = \int \overline{\Pi}(\lambda) d\lambda$ where the measure $d\lambda$ accommodates both continuous and discrete spectra and $\overline{\Pi}(\lambda)$ is the projection operator that projects onto the state space spanned by the eigenstates of $i[H_F, H_B]$ with eigenvalue λ. The degeneracy of the eigenvalue λ is given by the density function $\rho(\lambda) = \mathrm{Tr}[\overline{\Pi}(\lambda)]$ where Tr is the trace. To emphasize the degeneracy of λ we write the identity operator as

$$\mathbf{1} = \int \rho(\lambda) \Pi(\lambda) d\lambda \tag{9}$$

where $\Pi(\lambda) = \overline{\Pi}(\lambda)/\rho(\lambda)$ is an operator with unit trace. Thus, for example, $i[H_F, H_B] = \int \lambda \rho(\lambda) \Pi(\lambda) d\lambda$. Multiplying (8) on the right by (9) and ignoring the term of order τ^3 gives

$$S_{m,n} = U_B(m\tau) U_F(n\tau) \int I_{m,n}(\lambda) \rho(\lambda) \Pi(\lambda) d\lambda \tag{10}$$

where

$$I_{m,n}(\lambda) = \sum_{v=0}^{m} \cdots \sum_{\ell=0}^{s} \sum_{k=0}^{\ell} \exp\left[-i\,(v + \cdots + \ell + k)\tau^2 \lambda\right] \quad . \tag{11}$$

In the following analysis terms of order τ^3 are negligible and are ignored. Performing the algebraic manipulations described in Appendix B eventually yields

$$I_{m,n}(\lambda) = \frac{\prod_{q=0}^{n-1} \{\, 1 - \exp[-i(n+m-q)\tau^2 \lambda]\}}{\prod_{q=1}^{n} [1 - \exp(-iq\tau^2 \lambda)]} \ . \tag{12}$$

There are three important points to be made about the results in Eqs. (10) and (12). The first is that if the universe's origin state $|\psi_0\rangle$ is a $\lambda = 0$ eigenstate of the operator $i\,[H_F,\, H_B]$ then the integral in (10) can effectively be replaced with $I_{m,n}(0)\rho(0)\Pi(0)$ and the resulting evolution is equivalent to the T invariant case. In order to focus on the implications of T violation we assume that the universe's origin state $|\psi_0\rangle$ has a zero overlap with all $\lambda = 0$ eigenstates, i.e.

$$\Pi(0)|\psi_0\rangle = 0 \ . \tag{13}$$

We call this the *nonzero eigenvalue* condition for convenience. We look at the implications of relaxing this condition in Section 5. The second point is a useful symmetry property of $I_{m,n}(\lambda)$. Multiplying both the numerator and denominator of the right side of (12) by $\prod_{k=1}^{n+m} [1 - \exp(-ik\tau^2 \lambda)]$, canceling like terms and judiciously relabeling indices yields the symmetry

$$I_{m,n}(\lambda) = I_{n,m}(\lambda) \quad . \tag{14}$$

For the third point, it is straightforward to show for $n \leq m$ that the zeroes of the denominator of $I_{m,n}(\lambda)$ in (12) do not result in singularities, because if any factor $[1 - \exp(-iq\tau^2 \lambda)]$ in the denominator is zero for $\lambda = \lambda_0$, there is a corresponding factor $\{1 - \exp[-i(n+m-\ell)\tau^2 \lambda]\}$ in the numerator, where $\ell = n + m - jq$ and j is a positive integer, for which

$$\frac{1 - \exp[-i(n+m-\ell)\tau^2 \lambda]}{1 - \exp(-iq\tau^2 \lambda)} \to 1$$

as $\lambda \to \lambda_0$. The symmetry of $I_{m,n}(\lambda)$ in (14) means that the same result holds for the complementary case $n > m$.

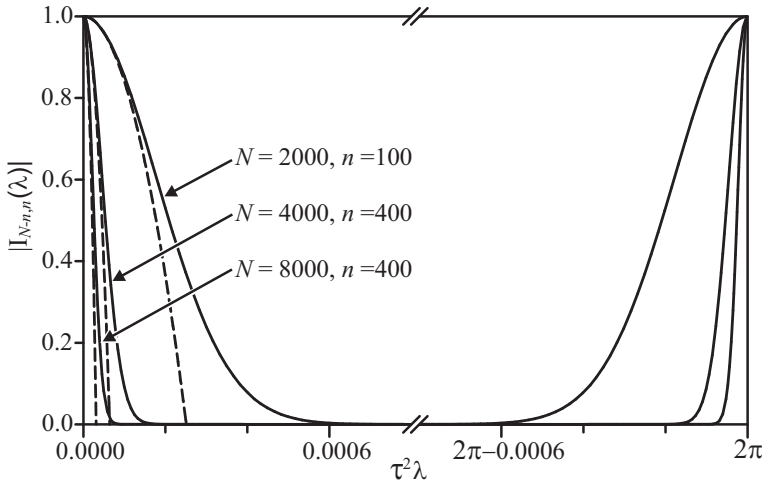

Fig. 2 Comparison of the function $|I_{N-n,n}(\lambda)|$ (solid curves) with the quadratic approximation (dashed curves) for various values of N and n. All curves have been scaled to give a peak value of 1.

3 Interference between paths

We now analyze the behavior of $S_{N-n,n}$ in (6) in some detail. Let $t_F = n\tau$ and $t_B = (N-n)\tau$ be the aggregate times that the universe evolves forwards and backwards, respectively, where N is the total number of steps and $t_{\text{tot}} = t_F + t_B = N\tau$ is the corresponding total time.

Certain features of $I_{N-n,n}(\lambda)$ are more apparent when its modulus $|I_{N-n,n}(\lambda)|$ is written as the product of diffraction grating interference functions, i.e.

$$|I_{N-n,n}(\lambda)| = \left| \prod_{q=1}^{n} \frac{\sin(\alpha_q \beta_q \tau^2 \lambda)}{\sin(\beta_q \tau^2 \lambda)} \right| ,$$

where $\alpha_q = \frac{1}{q}(N+1-q)$ and $\beta_q = \frac{q}{2}$. $|I_{N-n,n}(\lambda)|$ is a periodic function with a period of $2\pi/\tau^2$ and has maxima at $\lambda = k2\pi/\tau^2$ of $\Pi_{q=1}^{n}\alpha_q = \binom{N}{n}$ for integer k. For the times $t_F = n\tau$ and $t_B = (N-n)\tau$ to be physically measurable intervals, both n and $(N-n)$ need be very much larger than unity. In these cases we can treat $|I_{N-n,n}(\lambda)|$ as comprising narrow peaks at $\lambda = k2\pi/\tau^2$, for integer k, and being negligible elsewhere. Expanding $|I_{N-n,n}(\lambda)|$ as a power series in λ gives the quadratic approximation $|I_{m,n}(\lambda)| \approx \binom{N}{n}[1 - \frac{1}{24}n(N-n)(N+1)\tau^4\lambda^2]$. Fig. 2 illustrates the accuracy of the approximation near $\lambda = 0$. The zeroes of the quadratic approximation can be used as a reasonable estimate of the width $W_{N-n,n}$ of the peaks. This estimate is given by

$$W_{N-n,n} = \sqrt{\frac{24}{\tau^4 n(N-n)(N+1)]}} \quad .$$

To estimate typical values for the eigenvalues λ we take neutral kaon evolution as a prototypical T-violating process. We first consider the problem for a single kaon for which we label the eigenvalues and Hamiltonians with a superscript (1). The eigenvalues $\lambda^{(1)}$ of $i[H_F^{(1)}, H_B^{(1)}]$ in the 2 dimensional subspace spanned by the kaon and anti-kaon states can be found using the phenomenological model of Lee and Wolfenstein [16]. In Appendix C we show that this gives $\lambda^{(1)} \approx \pm 10^{17}\, s^{-2}$ (in units where $\hbar = 1$) using empirical values of Yao *et al.* [27].

Next, we estimate the eigenvalues of the operator $i[H_F, H_B]$ for the universe containing M kaons. For this we use the fact that the Hamiltonian of a non-interacting set of subsystems is simply the sum of the Hamiltonians of the subsystems. Assuming that the collection of kaons in the universe do not interact we can approximate each eigenvalue λ of the operator $i\,[H_F,\, H_B]$ as a sum $\lambda = \sum_j \lambda_j^{(1)}$ where $\lambda_j^{(1)}$ is an eigenvalue of $i[H_F^{(1)}, H_B^{(1)}]$ for the jth kaon. The values of λ range from $-M|\lambda^{(1)}|$ to $M|\lambda^{(1)}|$. The degeneracy of the eigenvalue $\lambda = k|\lambda^{(1)}|$ for integer k is given by the density function $\rho(\lambda) = M!/(\frac{1}{2}M - k)!(\frac{1}{2}M + k)!$. Note that $2^{-M}\rho(\lambda)$ is approximately a Gaussian distribution over λ with a mean of zero and standard deviation of $\lambda_{SD} = \frac{1}{2}\sqrt{M}|\lambda^{(1)}|$ and so we shall take the typical values of λ to range over $-\lambda_{\rm SD}, \ldots, \lambda_{\rm SD}$. Setting M to be a fraction, $f < 1$, of the average total number of particles in the universe, say 10^{80}, gives $M \approx f10^{80}$ and so $\lambda_{\rm SD} \approx \sqrt{f}10^{57}\, s^{-2}$.

There are two important limiting cases to consider that depend on the relative sizes of the width parameters $W_{N-n,n}$ and $\lambda_{\rm SD}$ of $I_{N-n,n}(\lambda)$ and $\rho(\lambda)$, respectively, as illustrated in Fig. 3. The first is the regime given by $W_{N-n,n} \gg \lambda_{\rm SD}$, i.e. relatively broad peaks of $I_{N-n,n}(\lambda)$. In this limit $I_{N-n,n}(\lambda)$ can be treated as approximately constant over the range $|\lambda| < \lambda_{\rm SD}$ and so the integral in (10) is well approximated by $\binom{N}{n}\mathbf{1}$ where $\mathbf{1}$ is the identity operator. This implies that there is negligible destructive interference between the paths represented by $\langle\phi|S_{N-n,n}|\psi_0\rangle$. The second case is the regime $W_{N-n,n} \ll \lambda_{\rm SD}$, i.e. relatively narrow peaks of $I_{N-n,n}(\lambda)$. In the extreme version of this, $I_{N-n,n}(\lambda)$ is zero except in the neighborhood of $\lambda = 0$. The integral in (10) therefore becomes $\binom{N}{n}\overline{\Pi}(0) \propto \Pi(0)$, where $\overline{\Pi}(0)$ is the projection operator which projects onto the zero eigenvalue subspace, and so $S_{N-n,n} = U_B[(N-n)\tau]U_F(n\tau)\binom{N}{n}\overline{\Pi}(0)$. But according to the nonzero eigenvalue condition in (13), the origin state of the universe has no overlap with any zero-eigenstate state of the commutator and so $S_{N-n,n}|\psi_0\rangle = 0$. In other words, the corresponding paths tend to undergo complete *destructive interference* in this regime, and make negligible contribution to the sum on the right side of (6).

The physical implications of these two limiting cases become apparent on considering both forwards and backwards aggregated intervals $t_F = n\tau$ and $t_B = (N-n)\tau$ to be very much larger than the Planck time τ , i.e. $n, (N-n) \gg 1$. In this regime the width of the peaks of $I_{N-n,n}(\lambda)$ can be approximated as

$$W_{N-n,n} = \frac{\sqrt{24}}{\sqrt{\tau^4 n(N-n)N}}[1 + \mathcal{O}(\frac{1}{N})] \approx \frac{\sqrt{24}}{\tau^{1/2}\sqrt{t_F t_B t_{\rm tot}}}$$

where $t_{\rm tot} = t_F + t_B = N\tau$ is the total time. Complete destructive interference occurs for $W_{N-n,n} \ll \lambda_{\rm SD}$ which implies that $(t_F t_B t_{\rm tot})^{1/3} \gg \tau^{-1/3}(\lambda_{\rm SD})^{-2/3}$. For $t_F \sim t_B$ this implies

$$t_{\rm tot} \gg \frac{1}{\tau^{1/3}(\lambda_{\rm SD})^{2/3}} \approx f^{-1/3}10^{-23}s \quad .$$

The destructive interference means that the corresponding set of paths do not contribute to the sum on the right side of (6); these paths are effectively "pruned" from the set possible paths that the universe can take.

It is the condition $t_F \sim t_B \gg \tau$ that is responsible for this destructive interference. So to find paths that survive the interference we need to consider times $t_F \ll t_B$ or $t_B \ll t_F$ for total times $t_{\rm tot} = N\tau$ which are at least measurable. For example, consider total

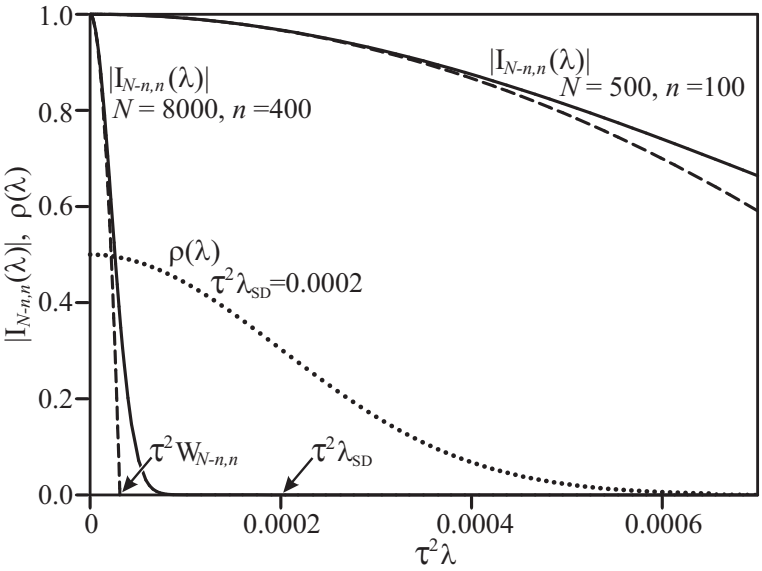

Fig. 3 Comparison of the density function $\rho(\lambda)$ (dotted curve) with the integrand function $|I_{N-n,n}(\lambda)|$ (solid curve) and its approximation (dashed curve). The curves representing $|I_{N-n,n}(\lambda)|$ and its approximation have been scaled to give a peak value of 1, whereas for clarity, the curve representing $\rho(\lambda)$ has been scaled to give a peak value of 0.5. The standard deviation of the density function is given by $\lambda_{\rm SD} = 0.0002/\tau^2$. The curve $|I_{N-n,n}(\lambda)|$ for $N = 500$ and $n = 100$ has a relatively broad peak whereas for $N = 8000$ and $n = 400$ the peak is relatively narrow compared to the peak in the density function curve.

times at the current resolution limit in time metrology [17] in which case $t_{\rm tot} \approx 10^{-17}s$ and so $N \approx 10^{27}$. Clearly there is no destructive interference for the cases $n = 0$ or $n = N$ (because there is only one path in each case). Also for $n \approx 1$ and $n \approx N-1$ the estimated width of the peaks of $I_{N-n,n}(\lambda)$ are given by

$$W_{N-n,n} = \frac{\sqrt{24}}{\tau^2 N}[1 + \mathcal{O}(\frac{1}{N})] \quad . \tag{15}$$

In these cases $W_{N-n,n} \approx 10^{61}s^{-2} \gg \lambda_{\rm SD}$ which implies *negligible* destructive interference. This means that the sum in (6) includes a number of paths, however, the paths are characterized by one of either t_F or t_B being negligible compared to the total time of $10^{-17}s$.

Finally consider the value of $t_{\rm tot}$ for which there is complete destructive interference for the sets of paths represented by $\langle\phi|S_{N-n,n}|\psi_0\rangle$ for all values of n except for $n = 0$ and $n = N$. In other words there is complete destructive interference for $n = 1, 2, \ldots (N-1)$ and so (6) is simply $|\Psi(N\tau)\rangle = S_{N,0}|\psi_0\rangle + S_{0,N}|\psi_0\rangle$. We only need to find the value of $t_{\rm tot}$ for the case $n = 1$ as this also ensures the destructive interference in the remaining cases. Thus we want to know the value of $t_{\rm tot}$ that satisfies $W_{N-1,1} \ll \lambda_{\rm SD}$. The width $W_{N-1,1}$ is approximated by (15) and so we have $\sqrt{24}/(\tau^2 N) \ll \lambda_{\rm SD}$ which implies $t_{\rm tot} \gg f^{-1/2}10^{-13}s$. In this regime, the two paths that survive the destructive interference represent exclusively forwards and exclusively backwards evolution. Figure 4 shows the effect of the destructive interference on the binary tree representation of the evolution.

In summary, for total times $t = N\tau > 10^{-17}s$ (6) can be replaced with $|\Psi(t)\rangle = [\sum_{n\approx 0} S_{N-n,n} + \sum_{n\approx N} S_{N-n,n}]|\psi_0\rangle$ and the integral in (10) is effectively the identity operator. We note that for $n \approx 0$, (10) then implies that $S_{N-n,n} = U_B(N\tau) + \mathcal{O}(\tau)$.

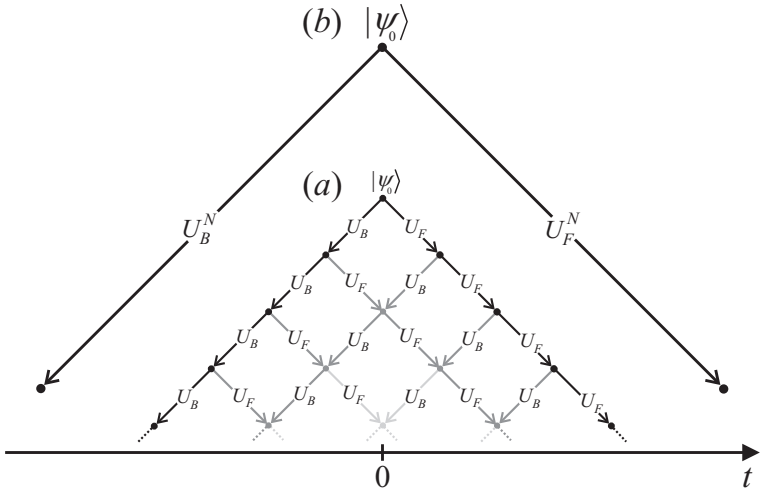

Fig. 4 The "pruning" of the binary tree in Fig. 1 by destructive interference for the case $H_F \neq H_B$ and $N \gg 1$. The relative weighting of each path segment is depicted by the greyness of the corresponding arrow with black and white representing maximal and minimal weighting, respectively: (a) shows an expanded view of the detail near the root node and (b) shows the whole tree on a much coarser scale.

Similarly, it can be shown using the symmetry property (14) that $S_{N-n,n} = U_F(N\tau) + \mathcal{O}(\tau)$ for $n \approx N$. On ignoring the terms of order τ we arrive at a key result of this paper, the *bievolution equation of motion*

$$|\Psi(t)\rangle = [U_B(t) + U_F(t)]\ |\psi_0\rangle\ . \tag{16}$$

The term "bievolution" here refers to the dual evolution generated by the two different Hamiltonians. The approximations made in deriving this equation become exact in the limit $t \gg f^{-1/2}10^{-13}s$.

4 Unidirectionality of time and standard quantum theory

Consider a situation where the Hamiltonians H_F and H_B leave distinguishable evidence in the state of the universe. For example, imagine that the universe's state at (total) time t_1 is

$$|\Psi(t_1)\rangle = |\phi\rangle + |\theta\rangle$$

where $|\phi\rangle = U_B(t_1)|\psi_0\rangle$ and $|\theta\rangle = U_F(t_1)|\psi_0\rangle$. Let $U_\mu(t_1 + a) = U_\mu(a)U_\mu(t_1)$ and $U_\mu(a)|x\rangle = |x_\mu\rangle$ where μ is either F or B, x is either ϕ or θ, and a is a suitable fixed time interval. Then, according to (16)

$$\begin{aligned}|\Psi(t_1 + a)\rangle &= U_B(t_1 + a)|\psi_0\rangle + U_F(t_1 + a)|\psi_0\rangle \\ &= U_B(a)U_B(t_1)|\psi_0\rangle + U_F(a)U_F(t_1)|\psi_0\rangle \\ &= U_B(a)|\phi\rangle + U_F(a)|\theta\rangle \\ &= |\phi_B\rangle + |\theta_F\rangle.\end{aligned}$$

The Hamiltonians leave *distinguishable* evidence in the state of the universe if they generate orthogonal states in the sense that $\langle\theta_B|\theta_F\rangle = \langle\phi_B|\phi_F\rangle = 0$. We assume this to be the case. Next, let this process occur twice such that $U_\nu(a)U_\mu(a)|x\rangle = |x_{\mu,\nu}\rangle$ and $U_\mu(t_1 + 2a) = U_\mu(a)U_\mu(t_1 + a)$ where μ and ν are each either F or B. We find, again by (16), that

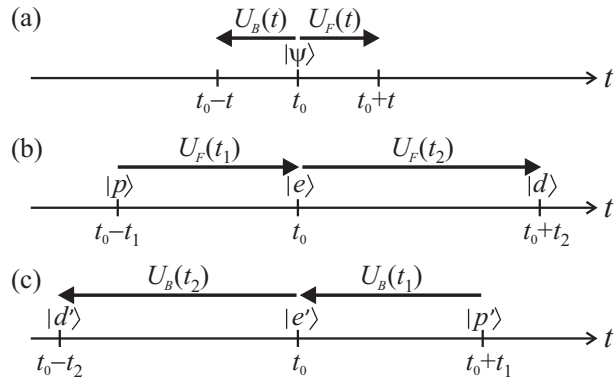

Fig. 5 The actions of $U_F(t)$ and $U_B(t)$ on various states of the universe in relation to clock time t. Diagram (a) represents the actions of $U_F(\tau)$ and $U_B(\tau)$ on an arbitrary state $|\psi\rangle$ at clock time t_0. Diagrams (b) and (c) represent the description of the evolution of the universe by an observer who finds evidence that the Hamiltonian is H_F or H_B, respectively.

$$\begin{aligned}|\Psi(t_1+2a)\rangle &= U_B(t_1+2a)|\psi_0\rangle + U_F(t_1+2a)|\psi_0\rangle \\ &= U_B(a)U_B(t_1+a)|\psi_0\rangle + U_F(a)U_F(t_1+a)|\psi_0\rangle \\ &= U_B(a)|\phi_B\rangle + U_F(a)|\theta_F\rangle \\ &= |\phi_{B,B}\rangle + |\theta_{F,F}\rangle \end{aligned} \tag{17}$$

which shows that *corroborating* evidence is left in each term of the superposition. Recall that the opposite signs in the exponents of the definitions of U_F and U_B in Eqs. (1) and (2) imply that they represent time evolution in opposing directions of time, as illustrated by Fig. 5(a). We conclude that the expressions $U_F(t)|\psi\rangle$ and $U_B(t)|\psi\rangle$ represent the universe evolving in opposite directions of time and in each case the evolution leaves corroborating evidence of the associated Hamiltonian in the state of the universe.

To interpret what these results mean, consider an observer within the universe who performs measurements to determine the nature of the Hamiltonian. According to (17) the observer would be in a superposition of two states, one representing the observer consistently obtaining physical evidence that the Hamiltonian is H_F, and the other representing the observer consistently obtaining physical evidence that the Hamiltonian is H_B. This means that each term on the right side of the bievolution equation (16) represents the observer having access to *only one version* of the Hamiltonian. It follows that the observer would describe the universe as evolving according to either

$$|\Psi(t)\rangle = U_F(t)\,|\psi_0\rangle \tag{18}$$

or

$$|\Psi(t)\rangle = U_B(t)\,|\psi_0\rangle \tag{19}$$

depending on whether the observer has evidence that the Hamiltonian is H_F or H_B, respectively. Each of (18) and (19) is the solution of the conventional Schrödinger equation for the corresponding version of the Hamiltonian. The observer's description would therefore be in accord with our own experience of the dynamics of the universe.

It is worth remarking that there is no inherent preferred direction of time in conventional quantum mechanics. Indeed, in conventional quantum theory the Hamiltonian is the generator of time translations in either direction of time. Here, however, each of (18) and (19) is associated with a particular direction of time evolution in the sense that $U_F(\tau)$ evolves the state of the universe along the positive$-t$ ("forward") direction and

$U_B(\tau)$ evolves the state of the universe along the negative$-t$ ("backward") direction, as illustrated in Figs. 5(b) and 5(c). This association stems from H_F being the generator of time translations *exclusively* in the positive$-t$ direction and H_B being the generator of time translations *exclusively* in the negative$-t$ direction in accord with Wigner's time reversal operation [12] as discussed in Section 2. The association has important consequences for the order that events occur in time. For example, consider the case where the observer finds evidence that the version of the Hamiltonian is H_F as shown in Fig. 5(b). Time evolution in this case is in the positive$-t$ direction. Imagine that a particular atom is undergoing spontaneous radiative decay at $t = t_0$ and let $|e\rangle$ represent the universe with the atom in an excited energy eigenstate. The state $|d\rangle = U_F(t_2)|e\rangle$ for an appropriate value of t_2 would represent the universe with the atom having decayed to a lower energy eigenstate along with its emitted radiation, say. The state $|p\rangle$, where $|e\rangle = U_F(t_1)|p\rangle$, could represent a stage in the preparation of the excited atom. The order that the preparation stage, excited atom and decay atom appear in time is dictated by the direction of the time evolution. The converse case where the observer finds evidence that the Hamiltonian is H_B is illustrated in Fig. 5(c) for which $|d'\rangle$, $|e'\rangle$ and $|p'\rangle$ represent the decayed atom, the excited atom, and the preparation stage, respectively. The reversed ordering of the events in this case is due to the opposite direction of time evolution that is associated with the H_B version of the Hamiltonian. In both cases the observer would find physical evidence of a fixed direction of time evolution and that direction would correspond to the version of the Hamiltonian observed.

We have allowed the universe to take any path through time according to (5). But provided there are sufficient T violation processes, destructive interference between paths will result in the universe evolving according to the bievolution equation (16). We have shown that in this case an observer within the universe would find physical evidence of only one version of the Hamiltonian. As a consequence, the observer would describe the universe as evolving according to the Schrödinger equation in the direction of time that corresponds to the observed version of the Hamiltonian. We have therefore demonstrated the *consistency between the bievolution equation and conventional quantum theory for an observer within the universe* and how the *unidirectional nature of time arises from the perspective of the observer.*

5 Eigenstates of $i[H_F, H_B]$ with zero eigenvalues

We now temporarily relax the nonzero eigenvalue condition in (13) and set the origin state to be an eigenstate of the commutator $i[H_F, H_B]$ with eigenvalue zero, i.e. we set

$$\overline{\Pi}(0)|\psi_0\rangle = |\psi_0\rangle \ .$$

As previously mentioned, the integral in (10) in this case can effectively be replaced with $I_{m,n}(0)\rho(0)\Pi(0) = \binom{n+m}{n}\overline{\Pi}(0)$ and so (6) becomes

$$|\Psi(N\tau)\rangle = \sum_{n=0}^{N} U_B[(N-n)\tau]U_F(n\tau)\binom{N}{n}|\psi_0\rangle$$

which represents all 2^N possible paths through time. We now reapply the analysis of the previous section for the processes $U_F(a)$ and $U_B(a)$ with $t_1 = N\tau$ and $a = k\tau$. There

is no destructive interference in this case, and so the state of the universe at the total time $t_1 + 2a$ is, according to (5),

$$|\Psi(t_1 + 2a)\rangle = [U_B(\tau) + U_F(\tau)]^{2k} \left[\sum_{n=0}^{N} U_B[(N-n)\tau] U_F(n\tau) \binom{N}{n} \right] |\psi_0\rangle$$

which represents 2^{N+2k} paths through time. These paths include the operator combinations $U_F(a)U_F(a)$, $U_F(a)U_B(a)$, $U_B(a)U_F(a)$, $U_B(a)U_B(a)$ (among many others) and so any distinguishable evidence of the Hamiltonians left in the state of the universe is not necessarily corroborated over repetitions of the process. This is not in accord with present experimental evidence. Moreover, consider the origin state of the most general form

$$|\psi_0\rangle = |\psi_0^{\perp}\rangle + |\psi_0^{\parallel}\rangle$$

where $\overline{\Pi}(0)|\psi_0^{\perp}\rangle = 0$ and $\overline{\Pi}(0)|\psi_0^{\parallel}\rangle = |\psi_0^{\parallel}\rangle$. For this case the bievolution equation (16) is replaced with

$$|\Psi(t)\rangle = [U_B(t) + U_F(t)]\ |\psi_0^{\perp}\rangle + \sum_{n=0}^{N} U_B[(N-n)\tau] U_F(n\tau) \binom{N}{n} |\psi_0^{\parallel}\rangle \tag{20}$$

where $t = N\tau$. However, as we have already seen, experimental evidence suggests that the universe is not following the corresponding set of paths represented by the last term on the right side. In order for our system to provide a physical model of the *observable* universe, it is therefore sufficient to only consider origin states that satisfy the nonzero eigenvalue condition in (13). We assume this to be the case in the remainder of this work.

6 Schrödinger's equation for bievolution

We can construct the differential form of the bievolution equation of motion as follows. Increasing t in (16) by a relatively small time interval δt yields $|\Psi(t+\delta t)\rangle = [U_B(\delta t)U_B(t) + U_F(\delta t)U_F(t)]\,|\psi_0\rangle$ and so the rate of change of the state is given by

$$\frac{\delta|\Psi(t)\rangle}{\delta t} = [iH_B U_B(t) - iH_F U_F(t)]\ |\psi_0\rangle + \mathcal{O}(\delta t)$$

where $\delta|\Psi(t)\rangle = |\Psi(t+\delta t)\rangle - |\Psi(t)\rangle$. Taking the limit $\delta t \to \tau$ and ignoring a term of order τ gives another key result, the *Schrödinger equation for bievolution*

$$\frac{d|\Psi(t)\rangle}{dt} = \frac{d|\psi_F(t)\rangle}{dt} - \frac{d|\psi_B(t)\rangle}{dt}$$

where $(d/dt)|\psi_\mu(t)\rangle = -iH_\mu|\psi_\mu(t)\rangle$ and $|\psi_\mu(t)\rangle = U_\mu(t)|\psi_0\rangle$ for $\mu = F$ or B.

7 Discussion

To gauge the full impact of T violation processes, it is instructive to compare the above analysis with a universe which obeys T invariance. In this case $H_F = H_B = H$ and it can be shown that

$$S_{N-n,n} = \exp[i(N-2n)\tau H] \binom{N}{n} \tag{21}$$

which is never zero and so all possible paths are included in (6). The direction of time is ambiguous as there is no physical evidence of any kind to single out one direction over the other. The fact that this ambiguity is removed for a universe with T violation processes leads directly to the proposition that *T violation processes are responsible for the phenomenological unidirectionality of time that we observe in the universe.*

Just as important is the implicit assertion here that *in the absence of a unidirectional nature, time is randomly directed at any given instant.* We allowed for the possibility of a random direction when we added amplitudes in (3). In hindsight, the analysis of the preceding sections can be seen to be a study of the dichotomy between randomly directed and unidirectional time evolution. In this context the important question about the unidirectional nature of time is not how one direction of time is chosen over the other, but rather how each direction is maintained consistently from one instant to the next. This is one of the underlying questions that is being addressed in this article, with T violation providing a possible mechanism as the answer.

We can also elaborate on what the time reversal operation means in the presence of T violating processes. The bievolution equation (16) shows that as the value of t increases, the states $U_F(t)|\psi_0\rangle$ and $U_B(t)|\psi_0\rangle$ in each branch of the superposition trace out respective trajectories in the state space. Decreasing values of t then simply "back tracks" the states along their respective trajectories in a consistent manner, essentially tracking back over what might be called their "histories". In contrast, the time reversal operation essentially interchanges $U_F(t)$ and $U_B(t)$ in (16). This implies that the time reversal operation refers to *switching between the branches of the bievolution equation* as opposed to tracking back over histories.

These results significantly elevate the importance of the matter-antimatter arrow in the study of the nature of time. The underlying T violation is based on the empirical evidence that different versions of the Hamiltonian operate for different directions of time evolution. Moreover, as discussed in Section 4, each direction of time can be uniquely identified by physical evidence left by the Hamiltonian. This is quite different to other arrows of time. In particular, the thermodynamic arrow does not distinguish between the different directions of time evolution in this way because entropy increases in the direction of time evolution *regardless of the direction.* This suggests that T violation plays a far more significant role in determining the nature of time than previously imagined.

We should mention that the physical implications of T violation described here depend the magnitudes of the eigenvalues λ. Our estimate of these eigenvalues depends on the number of particles involved in T violations processes and this has been fixed only at $f10^{80}$ where $f < 1$ is a parameter of undetermined value. However, our analysis does not depend critically on the value of f and, indeed, a large range of its values support the key results. Also, our analysis has focused on relatively large time scales in order to draw out the main consequences of T violation for the unidirectionality of

time. In doing so we have ignored terms of order τ, the Planck time. These terms imply that the directionality of time has finer details for relatively small time intervals. The elaboration of these details is, however, beyond the scope of this article.

There are potentially important experimental implications of the analysis presented here. A sufficiently small value of f would produce incomplete destructive interference and a deviation from the dynamics described by the bievolution equation (16). This would be manifest in a lower bound to the accuracy of time metrology. A more extreme case occurs in the very early universe during the inflation and radiation-dominated periods [18,19]. Baryogenesis would not have yet occurred and CP and T violation would have been relatively rare events. This implies that the value of f would have been negligible during these periods and so the direction of time evolution would have been uncertain. This has potentially important ramifications for cosmological models and their testing against observational data of the cosmic microwave background radiation [18].

The analysis presented here may also have important consequences for quantum gravity. Consider again the situation where $H_F = H_B = H$. Instead of expanding (5) into terms of the kind in (21) we write $U_F + U_B = 2\cos(H\tau)$ and so

$$|\Psi(N\tau)\rangle = 2^N \cos^N(H\tau)|\psi_0\rangle \ . \tag{22}$$

Let the eigenbasis of H be given by

$$H|E_n, \lambda\rangle = E_n|E_n, \lambda\rangle \ , \tag{23}$$

where E_n are the energy eigenvalues with $E_0 = 0$ and λ indexes different eigenstates in degenerate manifolds. Expanding $|\psi_0\rangle$ in this basis gives

$$|\Psi(N\tau)\rangle = 2^N \sum_{n,\lambda} c_{n,\lambda} \cos^N(E_n\tau)|E_n, \lambda\rangle \tag{24}$$

where $c_{n,\lambda} = \langle E_n, \lambda|\psi_0\rangle$. Let $|\psi_0\rangle$ be a state of bounded energy to the extent that $c_{n,\lambda} = 0$ for $|E_n| \geq 1/\tau$. Here $1/\tau$ is the Planck energy in units where $\hbar = 1$. This bound would be reasonable for a very young and very small universe. In this case, provided $c_{0,\lambda} \neq 0$ for some values of λ, the state $|\Psi(N\tau)\rangle$ becomes proportional to $\sum_\lambda c_{0,\lambda}|E_0, \lambda\rangle$ as $N \to \infty$. Hence in the limit as $N \to \infty$

$$H|\Psi(N\tau)\rangle = 0 \ . \tag{25}$$

According to (5), $|\Psi(N\tau)\rangle$ is a superposition of states each of which represents a net time ranging from $-N\tau$ to $+N\tau$. It therefore represents the whole *history* of the universe. (25) is analogous to the topological invariance condition discussed by Misner [20] in relation to the quantization of gravity. Topological invariance is needed in quantum gravity to maintain the invariance of physical quantities to transformations of spacetime manifolds. To make a direct connection with Misner's work, we let H represent the Hamiltonian $H(x)$ for gravitational and matter fields defined on a four dimensional spacetime manifold x, $\{\sigma_k\}$ represent a set of spacelike hypersurfaces indexed by integer k and separated by timelike intervals of τ, $U_F(n\tau)|\psi_0\rangle$ represent the state functional ψ_{σ_n} and $U_B(n\tau)|\psi_0\rangle$ represent the state functional $\psi_{\sigma_{-n}}$ for positive integer n, where ψ_σ is a functional of the gravitational and matter fields at the hypersurface σ. In this

representation $|\Psi(N\tau)\rangle$ represents a state functional which is a superposition of ψ_{σ_n} for $n = -N$ to N. (25) then represents the result that *topological invariant states are attractors of time symmetric evolution*. The zero energy eigenstate implies that the positive energy of matter fields is balanced by the negative energy of the gravitational potential energy [21,22,23].

This means that (25) corresponds to the Hamiltonian constraint of the Wheeler-DeWitt equation [24,25]. The Wheeler-DeWitt equation represents the entire history of the universe with no bias towards either direction of time. Here time symmetric evolution is unbiased in the same way in the sense that it represents evolution without a fixed direction of time and the state in (25) represent the entire history of the universe. These common features suggest that there is an elemental relationship between the Wheeler-DeWitt description and the present analysis. The present analysis may therefore offer a means of incorporating the two versions of the matter Hamiltonian in a Wheeler-DeWitt description of the universe for situations where $H_F \neq H_B$. Moreover, as the present analysis shows how the phenomenological unidirectionality of time arises when $H_F \neq H_B$, it may also shed fresh light on the problem of how time is established in the timeless Wheeler-DeWitt description.

In conclusion, the physical significance of the relatively weak and rare T violation processes exhibited by mesons has been a puzzle for more than four decades. Despite representing a fundamental time asymmetry, these processes have been regarded as having an insignificant affect on the physical nature of time. This has been due in part to the lack of a formalism for accommodating their time asymmetry in a single dynamical law. We have removed this obstacle by deriving, from a first principles analysis based on Feynman's sum over histories, a general method for incorporating both versions of the Hamiltonian for a T violating process, one for forwards and the other for backwards evolution, in a single dynamical equation of motion. Moreover we have shown that these processes can affect the time evolution of the universe on a grand scale. Indeed, we have shown that T violation provides, at least in principle, a physical mechanism for the phenomenological unidirectionality of time. Finally, the analysis presented here may also have important ramifications for time metrology, quantum gravity and the early development of the universe. At the very least it gives a clue to the quantum nature of time itself.

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Appendix A: Reordering factors of $S_{m,n}$

The operator $S_{m,n}$ is defined recursively in (7) as

$$S_{m,n} = \sum_{k=0}^{m} S_{m-k,n-1} U_F(\tau) U_B(k\tau) \tag{A.1}$$

with $S_{m,0} = U_B(m\tau)$ and so

$$S_{m,1} = \sum_{k=0}^{m} S_{m-k,0} U_F(\tau) U_B(k\tau) = \sum_{k=0}^{m} U_B[(m-k)\tau] U_F(\tau) U_B(k\tau) \ . \tag{A.2}$$

The Zassenhaus formula [26]

$$e^{A\epsilon}e^{B\epsilon}e^{-\frac{1}{2}\epsilon^2[A,B]}e^{\mathcal{O}(\epsilon^3)} = e^{B\epsilon}e^{A\epsilon}e^{-\frac{1}{2}\epsilon^2[B,A]}e^{\mathcal{O}(\epsilon^3)}$$

(which is related to the Baker-Campbell-Hausdorff formula) where A and B are arbitrary operators and ϵ is a small parameter, can be written as

$$e^{A\epsilon}e^{B\epsilon} = e^{B\epsilon}e^{A\epsilon}e^{\epsilon^2[A,B]}e^{\mathcal{O}(\epsilon^3)}$$

which shows that

$$e^{-ij\tau H_F}e^{ik\tau H_B} = e^{ik\tau H_B}e^{-ij\tau H_F}e^{jk\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \ . \tag{A.3}$$

In terms of the operators $U_F(\tau) = e^{-i\tau H_F}$ and $U_B(\tau) = e^{i\tau H_B}$ this result reads

$$U_F(j\tau)U_B(k\tau) = U_B(k\tau)U_F(j\tau)e^{jk\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \tag{A.4}$$

or equivalently

$$U_B(k\tau)U_F(j\tau) = U_F(j\tau)U_B(k\tau)e^{-jk\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \ . \tag{A.5}$$

Hence

$$\begin{aligned}
S_{m,n} &= \sum_{k=0}^{m} S_{m-k,n-1}U_F(\tau)U_B(k\tau) \\
&= \sum_{k=0}^{m} S_{m-k,n-1}U_B(k\tau)U_F(\tau)e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{k=0}^{m}\sum_{j=0}^{m-k} S_{m-k-j,n-2}U_F(\tau)U_B(j\tau)U_B(k\tau)U_F(\tau)e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{k=0}^{m}\sum_{j=0}^{m-k} S_{m-k-j,n-2}U_F(\tau)U_B[(j+k)\tau]U_F(\tau)e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{k=0}^{m}\sum_{j=0}^{m-k} S_{m-k-j,n-2}U_F(2\tau)U_B[(j+k)\tau]e^{-j\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{k=0}^{m}\sum_{j=0}^{m-k} S_{m-k-j,n-2}U_B[(j+k)\tau]U_F(2\tau)e^{(j+2k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \ .
\end{aligned} \tag{A.6}$$

Setting $\ell = j + k$ and rearranging the order the terms are summed to eliminate j then yields

$$\begin{aligned}
S_{m,n} &= \sum_{\ell=0}^{m}\sum_{k=0}^{\ell} S_{m-\ell,n-2}U_B(\ell\tau)U_F(2\tau)e^{(\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{\ell=0}^{m} S_{m-\ell,n-2}U_B(\ell\tau)U_F(2\tau)e^{\ell\tau^2[H_F,H_B]}\sum_{k=0}^{\ell} e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{\ell=0}^{m}\sum_{r=0}^{m-\ell} S_{m-\ell-r,n-3}U_F(\tau)U_B[(\ell+r)\tau]U_F(2\tau)e^{\ell\tau^2[H_F,H_B]}\sum_{k=0}^{\ell} e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\
&= \sum_{\ell=0}^{m}\sum_{r=0}^{m-\ell} S_{m-\ell-r,n-3}U_B[(\ell+r)\tau]U_F(3\tau)e^{(r+2\ell)\tau^2[H_F,H_B]}\sum_{k=0}^{\ell} e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)}
\end{aligned} \tag{A.7}$$

Rearranging as before but with $j = \ell + r$ and eliminating r yields

$$\begin{aligned} S_{m,n} &= \sum_{j=0}^{m}\sum_{\ell=0}^{j} S_{m-j,n-3}U_B(j\tau)U_F(3\tau)e^{(j+\ell)\tau^2[H_F,H_B]}\sum_{k=0}^{\ell} e^{k\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\ &= \sum_{j=0}^{m} S_{m-j,n-3}U_B(j\tau)U_F(3\tau)e^{j\tau^2[H_F,H_B]}\sum_{\ell=0}^{j}\sum_{k=0}^{\ell} e^{(\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \ . \end{aligned} \tag{A.8}$$

Continuing in this way gives, after w uses of the recursive relation (A.1),

$$S_{m,n} = \sum_{v=0}^{m} S_{m-v,n-w}U_B(v\tau)U_F(w\tau)e^{v\tau^2[H_F,H_B]}\sum_{r=0}^{v}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell} e^{(r+\cdots+\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \tag{A.9}$$

in which there are w summations. Letting $w = n$ then yields

$$\begin{aligned} S_{m,n} &= \sum_{v=0}^{m} S_{m-v,0}U_B(v\tau)U_F(n\tau)e^{v\tau^2[H_F,H_B]}\sum_{r=0}^{v}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell} e^{(r+\cdots+\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\ &= \sum_{v=0}^{m} U_B(m\tau)U_F(n\tau)e^{v\tau^2[H_F,H_B]}\sum_{r=0}^{v}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell} e^{(r+\cdots+\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \\ &= U_B(m\tau)U_F(n\tau)\sum_{v=0}^{m}\sum_{r=0}^{v}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell} e^{(v+r+\cdots+\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \ . \end{aligned} \tag{A.10}$$

Hence we arrive at (8)

$$S_{m,n} = U_B(m\tau)U_F(n\tau)\sum_{v=0}^{m}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell} e^{(v+\cdots+\ell+k)\tau^2[H_F,H_B]}e^{\mathcal{O}(\tau^3)} \tag{A.11}$$

in which there are n summations.

Appendix B: Simplifying $I_{m,n}(\lambda)$

Consider the following manipulations of *nested summations* of the kind

$$\sum_{j=0}^{w}\sum_{\ell=0}^{j}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\ell+m+\cdots+n+k)\theta} \ . \tag{B.1}$$

We call the summations "nested" because the upper limit n of the last summation over k is the index of the second-last summation and so on. The nested property means that summations cannot be evaluated independently of each other. However it is possible to reduce the number of nested summations by performing certain operations which we now describe.

First, by changing the order in which the indices j and ℓ are summed, we find

$$\sum_{j=0}^{w}\sum_{\ell=0}^{j}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\ell+m+\cdots+n+k)\theta} = \sum_{\ell=0}^{w}\sum_{j=\ell}^{w}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\ell+m+\cdots+n+k)\theta} \tag{B.2}$$

next, by cyclically interchanging the indices in the order $j \to k \to n \to s \to \cdots \to m \to \ell \to j$ on the right-hand side, we get

$$\sum_{j=0}^{w}\sum_{\ell=0}^{j}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\ell+m+\cdots+n+k)\theta} = \sum_{j=0}^{w}\sum_{k=j}^{w}\sum_{\ell=0}^{j}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s} e^{i(k+j+\ell+m+\cdots+n)\theta} \tag{B.3}$$

and finally, bringing the sum over k to the extreme right on the right-hand side gives

$$\sum_{j=0}^{w}\sum_{\ell=0}^{j}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\ell+m+\cdots+n+k)\theta} = \sum_{j=0}^{w}\sum_{\ell=0}^{j}\sum_{m=0}^{\ell}\cdots\sum_{n=0}^{s}\sum_{k=j}^{w} e^{i(j+\ell+m+\cdots+n+k)\theta} \tag{B.4}$$

We can abbreviate this general summation property as

$$\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\cdots+s+n+k)\theta} = \sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=j}^{w} e^{i(j+\cdots+s+n+k)\theta} \ . \tag{B.5}$$

Consider the product

$$\left(e^{i\theta}+1\right)\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} = e^{i\theta}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} + \sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} \ . \tag{B.6}$$

Using (B.5) for the two summations in the first term on the right-hand side gives

$$\begin{aligned}\left(e^{i\theta}+1\right)\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} &= e^{i\theta}\sum_{n=0}^{s}\sum_{k=n}^{s} e^{i(n+k)\theta} + \sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} \\ &= \sum_{n=0}^{s}\sum_{k=n}^{s} e^{i(n+k+1)\theta} + \sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} \\ &= \sum_{n=0}^{s}\sum_{k=n+1}^{s+1} e^{i(n+k)\theta} + \sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(n+k)\theta} = \sum_{n=0}^{s}\sum_{k=0}^{s+1} e^{i(n+k)\theta}\end{aligned} \tag{B.7}$$

The two *nested* summations on the left-hand side have been reduced to two *un-nested* summations on the right-hand side. Similarly, using (B.5) again as well as (B.7) we find

$$\begin{aligned}\left(e^{i2\theta}+e^{i\theta}+1\right)\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(s+n+k)\theta} &= e^{i2\theta}\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(s+n+k)\theta} \\ &\quad + \left(e^{i\theta}+1\right)\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(s+n+k)\theta} \\ &= \sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=s}^{t} e^{i(s+n+k+2)\theta} + \sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{s+1} e^{i(s+n+k)\theta} \\ &= \sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=s+2}^{t+2} e^{i(s+n+k)\theta} + \sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{s+1} e^{i(s+n+k)\theta} \\ &= \sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{t+2} e^{i(s+n+k)\theta} \ .\end{aligned} \tag{B.8}$$

Here the *three* nested summations on the left-hand side have been reduced to *two* nested summations and one un-nested summation on the right-hand side. This implies that,

for example, $p+1$ nested summations can be reduced to p nested and one un-nested summations as follows:

$$\Big(\sum_{r=0}^{p} e^{ir\theta}\Big)\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\cdots+s+n+k)\theta}}_{p+1 \text{ nested sums}} = \Big(\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s} e^{i(j+\cdots+s+n)\theta}}_{p \text{ nested sums}}\Big)\sum_{k=0}^{w+p} e^{ik\theta} \ . \tag{B.9}$$

Moreover, by repeating this process once more, the p nested summations of the right-hand side can be further reduced to $p-1$ nested and one un-nested summations. Continuing in this way eventually leads to the following result

$$\begin{aligned}\prod_{q=0}^{p}\Big(\sum_{r=0}^{q} e^{ir\theta}\Big)\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n} e^{i(j+\cdots+s+n+k)\theta}}_{p+1 \text{ nested sums}} &= \prod_{q=0}^{p-1}\Big(\sum_{r=0}^{q} e^{ir\theta}\Big)\Big(\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s} e^{i(j+\cdots+s+n)\theta}}_{p \text{ nested sums}}\Big)\sum_{k=0}^{w+p} e^{ik\theta} \\ &= \underbrace{\Big(\sum_{j=0}^{w} e^{ij\theta}\Big)\cdots\Big(\sum_{n=0}^{w+p-1} e^{in\theta}\Big)\Big(\sum_{k=0}^{w+p} e^{ik\theta}\Big)}_{p+1 \text{ un-nested sums}} \\ &= \prod_{q=0}^{p}\Big(\sum_{j=0}^{w+q} e^{ij\theta}\Big) \ . \end{aligned} \tag{B.10}$$

Thus

$$\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n}}_{p+1 \text{ nested sums}} e^{i(j+\cdots+s+n+k)\theta} = \frac{\prod_{q=0}^{p}\Big(\sum_{j=0}^{w+q} e^{ij\theta}\Big)}{\prod_{q=0}^{p}\Big(\sum_{r=0}^{q} e^{ir\theta}\Big)} \ . \tag{B.11}$$

Evaluating the two geometric series on the right-hand side then gives

$$\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n}}_{p+1 \text{ nested sums}} e^{i(j+\cdots+s+n+k)\theta} = \frac{\prod_{q=0}^{p}\Big(1-e^{i(w+q+1)\theta}\Big)}{\prod_{q=0}^{p}\Big(1-e^{i(q+1)\theta}\Big)} = \frac{\prod_{q=1}^{p+1}\Big(1-e^{i(w+q)\theta}\Big)}{\prod_{q=1}^{p+1}\Big(1-e^{iq\theta}\Big)} \ . \tag{B.12}$$

Adjusting the number of nested summations of the left-hand side then gives the useful result that

$$\underbrace{\sum_{j=0}^{w}\cdots\sum_{s=0}^{t}\sum_{n=0}^{s}\sum_{k=0}^{n}}_{p \text{ nested sums}} e^{i(j+\cdots+s+n+k)\theta} = \frac{\prod_{q=1}^{p}\Big(1-e^{i(w+q)\theta}\Big)}{\prod_{q=1}^{p}\Big(1-e^{iq\theta}\Big)} \ . \tag{B.13}$$

We can now use this result to simplify the n nested summations in (11) as follows. First we note that

$$I_{m,n}(\lambda) = \sum_{v=0}^{m}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell} e^{-i(v+\cdots+\ell+k)\tau^2\lambda} = \frac{\prod_{q=1}^{n}\Big(1-e^{-i(m+q)\tau^2\lambda}\Big)}{\prod_{q=1}^{n}\Big(1-e^{-iq\tau^2\lambda}\Big)} \tag{B.14}$$

and then by redefining the index q in the numerator we arrive at

$$I_{m,n}(\lambda) = \frac{\prod_{q=0}^{n-1}\left(1 - e^{-i(n+m-q)\tau^2\lambda}\right)}{\prod_{q=1}^{n}\left(1 - e^{-iq\tau^2\lambda}\right)} \tag{B.15}$$

which is (12) of the paper.

Appendix C: Estimating eigenvalues

We estimate the eigenvalues of the commutator $[H_F^{(1)}, H_B^{(1)}]$ on the state space of a kaon field mode that represents a single particle. However, to avoid unnecessary clutter, we will omit the superscript "(1)" on the single particle Hamiltonians in the remainder of this section.

It is convenient to invoke the CPT theorem and use the CP operator instead of the T operator as follows

$$H_B = T\, H_F T^{-1} = (CP)^{-1}\, H_F\, (CP)\ . \tag{C.1}$$

Consider the matrix representation in the kaon state space where the vectors $(1,0)$ and $(0,1)$ represent the kaon and anti-kaon states $|K^0\rangle$ and $|\overline{K}^0\rangle$, respectively. In this basis the general form of the CP operator is

$$CP = \begin{bmatrix} 0 & e^{i\eta} \\ e^{i\xi} & 0 \end{bmatrix} \tag{C.2}$$

where η and ξ are possible phase angles. By redefining the complex phase of the kaon state where $|K^0\rangle$ is multiplied by $e^{i(\eta-\xi)/2}$, the CP operator becomes

$$CP = \begin{bmatrix} 0 & e^{i\theta} \\ e^{i\theta} & 0 \end{bmatrix} = e^{i\theta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{C.3}$$

where $\theta = \frac{1}{2}(\eta + \xi)$. (This is a more useful transformation for us here compared to the usual one, which multiplies $|K^0\rangle$ by $e^{-i\xi}$.) Let M_{ij} be the matrix elements of H_F in the kaon state basis where $M_{11} = \langle K^0|H_F|K^0\rangle$ etc. To find the corresponding matrix elements of H_F we use (C.1), that is

$$\begin{aligned} H_B &= (CP)^{-1}\, H_F\, (CP) \\ &= e^{-i\theta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} e^{i\theta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}\ . \end{aligned}$$

Hence we immediately find

$$\begin{aligned} H_F H_B &= \begin{bmatrix} M_{11}M_{22} + (M_{12})^2 & M_{11}(M_{21} + M_{12}) \\ M_{22}(M_{21} + M_{12}) & M_{11}M_{22} + (M_{21})^2 \end{bmatrix} \\ H_B H_F &= \begin{bmatrix} M_{11}M_{22} + (M_{21})^2 & M_{22}(M_{21} + M_{12}) \\ M_{11}(M_{21} + M_{12}) & M_{11}M_{22} + (M_{12})^2 \end{bmatrix} \quad , \end{aligned}$$

and so the commutator is

$$[H_F, H_B] = H_F H_B - H_B H_F = \begin{bmatrix} (M_{12})^2 - (M_{21})^2 & (M_{11} - M_{22})(M_{21} + M_{12}) \\ -(M_{11} - M_{22})(M_{21} + M_{12}) & -[(M_{12})^2 - (M_{21})^2] \end{bmatrix} \tag{C.4}$$

which is in the form

$$[H_F, H_B] = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix} . \tag{C.5}$$

The eigenvalues of the last matrix are $\pm\sqrt{A^2 - B^2}$. Note that A is imaginary and B is real, and so the eigenvalues are imaginary. We now determine empirical values for the variables $M_{i,j}$ in (C.4).

The phenomenological model given in Lee and Wolfenstein [16] describes the evolution of the amplitudes $a(t)$ and $b(t)$ for the single-particle kaon field to be in the states $|K^0\rangle$ and $|\overline{K}^0\rangle$, respectively, by the differential equation

$$\frac{d\psi(t)}{dt} = -\frac{1}{\hbar}(\Gamma + iM)\psi(t) \tag{C.6}$$

where $\psi = \binom{a}{b}$. The matrix on the right-hand side is defined by

$$\Gamma + iM = D + i(E_1\sigma_1 + E_2\sigma_2 + E_3\sigma_3) \tag{C.7}$$

where the matrices Γ and M are Hermitian and

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{C.8}$$

The analysis below shows that the matrix M in (C.6) represents a CP violating Hamiltonian. For our purposes, it provides a sufficient estimate of the T violating Hamiltonian for a neutral kaon. Accordingly, we use the empirical values of the elements of M as estimates of the corresponding matrix elements $M_{i,j}$ of the Hamiltonian H_F that appears in (C.4). We should also mention that we could use M as an estimate of the matrix elements of H_B instead, however the only difference this would make is that the matrix elements of the commutator $[H_F, H_B]$ would be multiplied by -1, and this has no physical consequences because the two eigenvalues are the negatives of each other.

If the CPT theorem holds, which we assume to be the case, then $E_1 = E\cos\phi$, $E_2 = E\sin\phi$ and $E_3 = 0$. Here ϕ is a complex parameter [16] which is related to the CP violation parameter ϵ by $\epsilon = (1 - e^{i\phi})/(1 + e^{i\phi})$, or equivalently,

$$e^{i\phi} = \frac{1 - \epsilon}{1 + \epsilon} . \tag{C.9}$$

The values of D and E comprise the decay rates γ_i and masses m_i of the short and long-lived components, $|K_1^0\rangle$ and $|K_2^0\rangle$ respectively, of $|K^0\rangle$ as follows

$$D = \tfrac{1}{4}(\gamma_1 + \gamma_2) + \tfrac{1}{2}i(m_1 + m_2) \tag{C.10}$$

$$iE = \tfrac{1}{4}(\gamma_1 - \gamma_2) + \tfrac{1}{2}i(m_1 - m_2) . \tag{C.11}$$

Here γ_i and m_i are the energies $\gamma_i = \hbar\gamma_i'$ and $m_i = m_i'c^2$, respectively, where γ_i' and m_i' are the actual decay rates and masses, respectively. Also note [27] that $|\epsilon| \approx 2.3 \times 10^{-3}$ and $\arg(\epsilon) \approx 45°$ and so from (C.9) we find $e^{i\phi} = 1 - 2\epsilon + \mathcal{O}(\epsilon^2)$. We can therefore make the following approximations

$$\begin{aligned} e^{i\phi} &= 1 - \sqrt{2}|\epsilon|(1+i) + \mathcal{O}(\epsilon^2) \ , \\ e^{i\phi^*} &= 1/(e^{i\phi})^* = 1 + \sqrt{2}|\epsilon|(1-i) + \mathcal{O}(\epsilon^2) \ , \\ e^{-i\phi} &= 1/(e^{i\phi}) = 1 + \sqrt{2}|\epsilon|(1+i) + \mathcal{O}(\epsilon^2) \ , \\ e^{-i\phi^*} &= \ (e^{i\phi})^* \ = 1 - \sqrt{2}|\epsilon|(1-i) + \mathcal{O}(\epsilon^2) \ . \end{aligned}$$

This allows us to write down the following useful results

$$\begin{aligned} \tfrac{1}{2}(e^{i\phi} + e^{i\phi^*}) &= 1 - i\sqrt{2}|\epsilon| + \mathcal{O}(\epsilon^2) \ , \\ \tfrac{1}{2}(e^{-i\phi} + e^{-i\phi^*}) &= 1 + i\sqrt{2}|\epsilon| + \mathcal{O}(\epsilon^2) \ , \\ \tfrac{1}{2}(e^{i\phi} - e^{i\phi^*}) &= -\sqrt{2}|\epsilon| + \mathcal{O}(\epsilon^2) \ , \\ \tfrac{1}{2}(e^{-i\phi} - e^{-i\phi^*}) &= \sqrt{2}|\epsilon| + \mathcal{O}(\epsilon^2) \ . \end{aligned}$$

From Eqs. (C.7), (C.8), (C.10) and (C.11) we find

$$\begin{aligned} \Gamma + iM &= D + i(E\cos\phi\ \sigma_1 + E\sin\phi\ \sigma_2) \\ &= \begin{bmatrix} \frac{1}{4}(\gamma_1+\gamma_2) + \frac{1}{2}i(m_1+m_2) & \Big[\frac{1}{4}(\gamma_1-\gamma_2) + \frac{1}{2}i(m_1-m_2)\Big]e^{-i\phi} \\ \Big[\frac{1}{4}(\gamma_1-\gamma_2) + \frac{1}{2}i(m_1-m_2)\Big]e^{i\phi} & \frac{1}{4}(\gamma_1+\gamma_2) + \frac{1}{2}i(m_1+m_2) \end{bmatrix} . \end{aligned} \tag{C.12}$$

The Hermitian transpose of this equation is

$$\Gamma - iM = \begin{bmatrix} \frac{1}{4}(\gamma_1+\gamma_2) - \frac{1}{2}i(m_1+m_2) & \Big[\frac{1}{4}(\gamma_1-\gamma_2) - \frac{1}{2}i(m_1-m_2)\Big]e^{-i\phi^*} \\ \Big[\frac{1}{4}(\gamma_1-\gamma_2) - \frac{1}{2}i(m_1-m_2)\Big]e^{i\phi^*} & \frac{1}{4}(\gamma_1+\gamma_2) - \frac{1}{2}i(m_1+m_2) \end{bmatrix} . \tag{C.13}$$

By subtracting (C.13) from (C.12) and dividing the result by $2i$ we get

$$\begin{aligned} M = &\begin{bmatrix} \frac{1}{2}(m_1+m_2) & -i\frac{1}{4}(\gamma_1-\gamma_2)\sqrt{2}|\epsilon| + \frac{1}{2}(m_1-m_2)(1+i\sqrt{2}|\epsilon|) \\ i\frac{1}{4}(\gamma_1-\gamma_2)\sqrt{2}|\epsilon| + \frac{1}{2}(m_1-m_2)(1-i\sqrt{2}|\epsilon|) & \frac{1}{2}(m_1+m_2) \end{bmatrix} \\ &+\mathcal{O}(\epsilon^2) \ . \end{aligned} \tag{C.14}$$

This shows that $M_{11} = M_{22}$ and so $B = 0$ in Eqs. (C.4) and (C.5). The eigenvalues of the commutator $[H_F, H_B]$ are therefore simply $\pm A$ where

$$\begin{aligned} A &= M_{12}^2 - M_{21}^2 \\ &= \Big[-i\tfrac{1}{4}(\gamma_1-\gamma_2)\sqrt{2}|\epsilon| + \tfrac{1}{2}(m_1-m_2)(1+i\sqrt{2}|\epsilon|)\Big]^2 \\ &\qquad -\Big[i\tfrac{1}{4}(\gamma_1-\gamma_2)\sqrt{2}|\epsilon| + \tfrac{1}{2}(m_1-m_2)(1-i\sqrt{2}|\epsilon|)\Big]^2 + \mathcal{O}(\epsilon^2) \\ &= -i\tfrac{1}{4}(\gamma_1-\gamma_2)\sqrt{2}|\epsilon|\tfrac{1}{2}(m_1-m_2)\Big[2(1+i\sqrt{2}|\epsilon|) + 2(1-i\sqrt{2}|\epsilon|)\Big] \\ &\qquad +\tfrac{1}{4}(m_1-m_2)^2\Big[(1+i\sqrt{2}|\epsilon|)^2 - (1-i\sqrt{2}|\epsilon|)^2\Big] + \mathcal{O}(\epsilon^2) \end{aligned}$$

$$
\begin{aligned}
&= i\Big[-(\gamma_1-\gamma_2)\sqrt{2}|\epsilon|\tfrac{1}{2}(m_1-m_2)+(m_1-m_2)^2\sqrt{2}|\epsilon|\Big]+\mathcal{O}(\epsilon^2)\\
&= i\sqrt{2}|\epsilon|\Big[(\gamma_1-\gamma_2)\tfrac{1}{2}(m_2-m_1)+(m_1-m_2)^2\Big]+\mathcal{O}(\epsilon^2)\\
&= i\sqrt{2}|\epsilon|\Big[\tfrac{1}{2}\Delta\gamma\Delta m+(\Delta m)^2\Big]+\mathcal{O}(\epsilon^2) \qquad \text{(C.15)}
\end{aligned}
$$

where $\Delta m = (m_2 - m_1)$ and $\Delta\gamma = (\gamma_1 - \gamma_2)$.

Finally, the empirical values [27]

$$
\begin{aligned}
\Delta m &\approx 0.56\times 10^{10}\hbar s^{-1}\ ,\\
\Delta\gamma &\approx 1.1\times 10^{10}\hbar s^{-1}\ ,
\end{aligned}
$$

show that the eigenvalue A in (C.15) can be approximated as

$$A \approx i10^{17}\hbar^2 s^{-2}\ . \qquad \text{(C.16)}$$

This is the value used in the paper for $\lambda^{(1)}$ where $\lambda^{(1)} = \pm iA$. Note that in the main text, we use units in which $\hbar = 1$. This is equivalent to replacing the Hamiltonians H_F and H_B in this section with $H_F/\hbar$ and $H_B/\hbar$, respectively. In that case the commutator in (C.5) has the form $[H_F, H_B]/\hbar^2$ which has a dimension of (time)^2. Hence $\lambda^{(1)} \approx \pm 10^{17}\, s^{-2}$.

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