

# Particle-wave duality: a dichotomy between symmetry and asymmetry

BY JOAN A. VACCARO

*Centre for Quantum Computation and Communication Technology (Australian Research Council), Centre for Quantum Dynamics, Griffith University, Brisbane, Queensland 4111, Australia*

---

Symmetry plays a central role in many areas of modern physics. Here we show that it also underpins the dual particle and wave nature of quantum systems. We begin by noting that a classical point particle breaks translational symmetry whereas a wave with uniform amplitude does not. This provides a basis for associating particle nature with asymmetry and wave nature with symmetry. We derive expressions for the maximum amount of classical information we can have about the symmetry and asymmetry of a quantum system with respect to an arbitrary group. We find that the sum of the information about the symmetry (wave nature) and the asymmetry (particle nature) is bounded by  $\log(D)$  where  $D$  is the dimension of the Hilbert space. The combination of multiple systems is shown to exhibit greater symmetry and thus more wavelike character. In particular, a class of entangled systems is shown to be capable of exhibiting wave-like symmetry as a whole while exhibiting particle-like asymmetry internally. We also show that superdense coding can be viewed as being essentially an interference phenomenon involving wave-like symmetry with respect to the group of Pauli operators.

## 1. Introduction

The duality of particle and wave nature is one of the tenets of modern quantum theory. Feynman summarised its importance by remarking that it contains the *only* mystery of quantum theory (Feynman *et al.* 1963). Often the duality is rephrased in terms of Bohr's complementarity principle (Bohr 1935) where particle nature is equated with well-defined position and wave nature with well-defined momentum. In the last few decades attempts have been made to quantify the duality more rigorously. For example Wootters and Zurek (1979) formulated an inequality for a double slit experiment that expresses a lower bound on the loss of path information for a given sharpness of the interference pattern. The first experimental realisation of distinct particle and wave properties of individual photons was demonstrated by Grangier *et al.* (1986) using two different experimental arrangements and a heralded single photon source. Scully *et al.* (1991) explored the erasure of path information and the recovery of an interference pattern using sub-ensembles conditioned on ancillary measurements. A debate regarding the application of an uncertainty principle

ensured (see Wiseman 1998 and references therein). Later Englert (1996) refined the mathematical representation of the duality by deriving an inequality for a two-way interferometer that limits the distinguishability of the outcomes of a path measurement and the visibility of the interference pattern and made the distinction between a priori predictability and distinguishability of the paths. Björk and Karlsson (1998) then extended the analysis to include quantum erasure. Barbieri *et al.* (2009) recently verified Englert's duality relationship experimentally. The study of the canonical position and momentum operators have been extended to general canonically conjugate observables (Pegg *et al.* 1990) and their properties explored using entropic uncertainty relations (Maassen & Uffink 1988, Rojas Gonzalez *et al.* 1995) and other measures (Luis 2003). The related study of the approximate simultaneous measurement of non-commuting observables has also a long history (Arthurs & Kelly 1965, Luis 2004, Ozawa 2004 and references therein). A different track has been to explore Bohr's complementarity principle in terms of the mutually unbiased bases (MUB) introduced by Schwinger (1960). The study of MUB is important for areas such as discrete Wigner functions (Gibbons *et al.* 2004), quantum error correction (Gottesman 1996), quantum cryptography (Miyadera & Imai 2006) as well as entangled systems (Kalev *et al.* 2009, Berta *et al.* 2010). Kurzynski *et al.* (2010) have recently examined the physical meaning of the operators associated with MUB for a spin-1 system.

Despite this work, there remain unexplored questions surrounding the mystery of the particle-wave duality. As pointed out by Englert (1996), the notions of wave and particle are borrowed from classical physics. An open question is whether each classical notion should be represented by a *single* quantum observable. In other words, can the problem be cast in terms of something more general such as a symmetry of the quantum system? Instead of looking for a pair of relevant observables could there not be a *set of observables or operators* associated with each notion? Moreover, the MUB approach to this problem yields complementary observables that tend to reflect mathematical properties of the underlying Hilbert space rather than objects of direct physical meaning (Kurzynski *et al.* 2010). This leads to the question of whether the particle-wave duality can be studied in a way that is *general and yet retains a consistent physical basis*. These are the key questions that we address in this paper.

We begin by showing in §2 that classical particle-like and wave-like properties have natural definitions in terms of a symmetry group. In §3 we differentiate between two sets of operations on the quantum system according to their effect on the symmetry of the system. This allows us to associate a set of operators that manipulate only particle-like properties (or asymmetry) of the system and another set that manipulate only the wave-like properties (or symmetry). We then define a convenient measure of the degree to which a state of the system exhibits particle or wave properties based on the ability to encode information in the system using the corresponding set of operations. A duality relation between the symmetry and asymmetry is derived in §4. In §5 we extend the analysis to composite systems. Applications of the formalism are given in §6 and we end with a discussion in §7. A preliminary version of this work was presented in Vaccaro 2006.

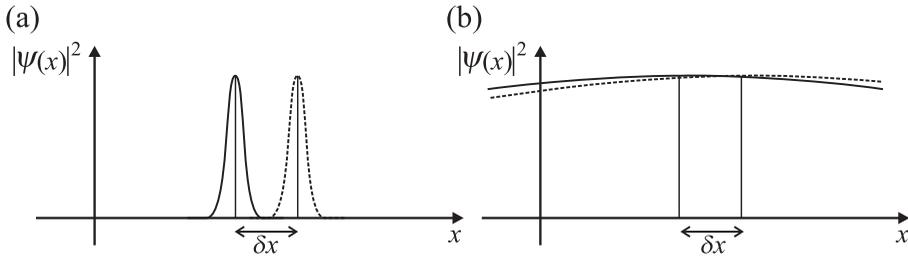


Figure 1. Spatial translations for (a) a narrow particle-like wave function and (b) a broad wave-like wave function. Solid curves represent the original functions and dashed curves the displaced versions for a translation of  $\delta x$  to the right. The wave functions are not normalised.

## 2. Symmetry of particles and waves

### (a) Particle-like and wave-like properties

Classical particles and waves respond in distinct ways to a spatial translation: the position of a classical point particle is displaced whereas the amplitude function of a uniform classical wave is invariant. These responses provide a basis for defining analogous wave and particle properties of quantum systems. Consider first a quantum system whose wave function comprises a relatively narrow peak in the position representation, as illustrated in figure 1(a). A spatial translation of  $\delta x$  along the  $x$  axis of sufficient magnitude can completely displace the system so that its wave function  $\psi(x)$  is mapped to an orthogonal wave function  $\psi(x - \delta x)$ . For example, the overlap  $\int \psi^*(x)\psi(x - \delta x)dx$  is negligible for Gaussian wave functions of the kind  $\psi(x) \propto \exp(-x^2/4\sigma^2)$  with  $\sigma \ll \delta x$ . Thus quantum systems with relatively narrow wave functions in the position representation behave as classical particles under spatial translations, as one would expect. Next, consider a system whose wave function is delocalised in the position representation so that the position probability density  $\text{Pr}(x) = |\psi(x)|^2$  is relatively “flat”, as illustrated in figure 1(b). Such wave functions are relatively invariant to spatial translations and so the system behaves as a classical wave under spatial translations. For example  $\text{Pr}(x - \delta x) \approx \text{Pr}(x)$  for Gaussian wave functions of the kind  $\psi(x) \propto \exp(-x^2/4\sigma^2)$  with  $\sigma \gg \delta x$ . Particle-like and wave-like properties of quantum systems can therefore be distinguished by whether the system is displaced or invariant, respectively, to spatial translations.

The invariance of a system to a given set of operations represents a particular symmetry of the system. In the case here wave-like properties represent the *symmetry* of a quantum system with respect to the group of spatial translations. Conversely, any displacement of a system under a spatial translation is a lack of this symmetry. Thus particle-like properties represent the *asymmetry* of a quantum system with respect to the group of spatial translations. In the following sections we generalise this concept by associating wave and particle properties with symmetry and asymmetry, respectively, for arbitrary groups.

*(b) Symmetry and interference*

However, we should not lose sight of the fact that for a system to be regarded as wave-like it must have an ability to produce interference in an interferometer of some kind. Symmetry, in contrast, is an intrinsic property of the system and can be quantified without reference to the details of any interferometer. We can, however, relate symmetry directly to interference in the following way. Taking the double slit experiment as the prototypical interferometer, we note that the positions of interference fringes on the screen can be changed by introducing a relative phase shift at the slits. This property allows an interferometer to be used as a communication channel between a sender at the slits and a receiver at the screen. The sender can encode a message by modifying the relative phase between the slits and the receiver can faithfully decode the message by observing the position of the fringes on the screen. In the next section we quantify the degree of symmetry of a system by its capacity to carry information in a likewise manner. In other words, an interferometer can be viewed as a communication channel and symmetry is measured in terms of information capacity. This information theoretic link between symmetry and interference applies to arbitrary systems. More will be said about this later.

Interference and symmetry share another common feature that is worth mentioning. Interference is only ever seen in a statistical sense and its full characterisation requires infinitely many observational events. For example, to see the wave-like character of an electron, we need a beam of electrons rather than a single one. This implies the wave-like character that we infer from interference has a similar statistical meaning. Likewise, the measure of symmetry that we use below is in terms of its capacity to carry information, and that capacity is derived by considering the statistics of random messages in the limit of infinitely many messages. So both have a statistical character and represent infinite ensembles. Nevertheless, we shall refer to the wave-like character and the symmetry of a single system.

Also while interference is typically viewed as occurring in coordinate space, there have been studies of interference in momentum space. For example, Rauch (1993) argued that in neutron interference experiments, when the path difference is sufficiently long that the neutron wave packets no longer overlap in space and the spatial interference has vanished, there can be persistent interference in the momentum representation in one of the output paths of the interferometer. In a similar vein, Pitaevskii and Stringari (1999) have shown that optically probing two spatially separated cold atomic gases can reveal interference in the momentum representation. Remarkably Pérez Prieto *et al.* (2001) have shown that in the collision of a wave packet with a potential barrier there can be suppression of a particular momentum value and enhancement of others due to the interference between the transmitted and reflected parts of the wave packet. Recently Ruschhaupt *et al.* (2009) took this a step further and devised a momentum-space interferometer using a trapped cold Bosonic atomic gas. Their proposal is to phase imprint part of the gas cloud using a detuned laser. The imprinting ensures that the wave function in the momentum representation

comprises a sum of two terms which interfere. Significantly, changing the amount of phase imprinting results in shifting the position of a node in the momentum distribution. These studies show that interference fringes may be present in the momentum representation and not the spatial representation, and in doing so they call for a broader interpretation of what constitutes wave-like character. But this does not pose a problem for us here. Indeed the arguments given above for symmetry and interference in configuration space also hold in momentum space provided that the symmetry in question is the invariance to momentum translations as opposed to spatial ones. In other words, wave-like character is ascribable to the symmetry of momentum space. This highlights the generality of our analysis in that it can capture the wave-like character of arbitrary symmetries.

(c) *Review of symmetry and asymmetry*

Before beginning our analysis in detail it will be useful to collect a number of relevant definitions and results. These relate to the symmetry and asymmetry of states with respect to a finite or compact Lie group. Let the group be  $G = \{g_1, g_2, \dots\}$  and have the unitary representation  $\{\hat{T}_g : g \in G\}$  on the system's Hilbert space  $H$ . We can borrow pertinent results about symmetry from the study of superselection rules (SSRs). A review of recent work on SSRs in the context of quantum information theory has been given by Bartlett *et al.* (2007). A state of the system is symmetric with respect to  $G$  if it satisfies

$$\mathcal{G}[\hat{\rho}] = \hat{\rho} \tag{2.1}$$

where the “twirl” of  $\hat{\rho}$  is defined as (Bartlett & Wiseman 2003)

$$\mathcal{G}[\hat{\rho}] \equiv \frac{1}{n_G} \sum_{g \in G} \hat{T}_g \hat{\rho} \hat{T}_g^\dagger, \tag{2.2}$$

$n_G$  is the order of  $G$  and  $\hat{\rho}$  is the system's density operator. Throughout this paper we use the notation for a finite group when referring to an arbitrary group. The equivalent results for a compact Lie group are easily found by appropriate modification of notation, for example, by replacing the averaged sum in equation (2.2) with an integral with respect to an invariant measure on the group. We shall call states satisfying equation (2.1) symmetric states. A symmetric state is unchanged by the actions of the group. Indeed equation (2.1) implies

$$\hat{T}_g \hat{\rho} \hat{T}_g^\dagger = \hat{T}_g \mathcal{G}[\hat{\rho}] \hat{T}_g^\dagger = \mathcal{G}[\hat{\rho}] = \hat{\rho} \tag{2.3}$$

for  $g \in G$ . In contrast, a state exhibits asymmetry with respect to  $G$  if

$$\hat{T}_g \hat{\rho} \hat{T}_g^\dagger \neq \hat{\rho} \tag{2.4}$$

for any  $g \in G$ . In this case we shall refer to  $\hat{\rho}$  as an asymmetric state or as having asymmetry with respect to  $G$ . A maximally-asymmetric pure state is a pure state for which  $\mathcal{G}[\hat{\rho}]$  is maximally mixed.

The degree to which a state is symmetric is given by the entropic measure (Vaccaro *et al.* 2008)

$$W_G(\hat{\rho}) = \log(D) - S(\mathcal{G}[\hat{\rho}]) . \tag{2.5}$$

where  $S(\hat{\varrho}) = -\text{Tr}[\hat{\varrho} \log(\hat{\varrho})]$  is the von Neumann entropy of  $\hat{\varrho}$  and  $D$  is the dimension of the system's Hilbert space. In the following we adopt the convention

of information theory and use the binary logarithm for  $\log(\cdot)$ . The measure  $W_G(\hat{\rho})$  has been called the *symmetry* of the state  $\hat{\rho}$  with respect to  $G$ . It has a maximum value of  $\log(D)$  for a symmetric pure state and a minimum of zero for a maximally-asymmetric pure state. Similarly, the *asymmetry* of  $\hat{\rho}$  with respect to  $G$  has been defined as the lack of symmetry as (Vaccaro *et al.* 2008)

$$A_G(\hat{\rho}) = S(\mathcal{G}[\hat{\rho}]) - S(\hat{\rho}) . \tag{2.6}$$

The asymmetry  $A_G(\hat{\rho})$  represents the ability of a system in state  $\hat{\rho}$  to act as a reference to alleviate the effects of the SSR.

It will be useful to have states with maximal asymmetry for exploring maximal particle-like properties. A system that is in a state with some asymmetry can be used as a reference to partially break the symmetry represented by the group  $G$ . For example, an object which lacks spherical symmetry can be used as a reference for orientation; the less spherical symmetry it has the better reference it is. Spherical symmetry is broken for systems that include the object. A quantum system can completely break the symmetry represented by the group  $G$  if its Hilbert space contains “reference states” for  $G$  of the form (Kitaev *et al.* 2004)

$$|g\rangle = \sum_{q,i,a} \sqrt{\frac{n_q}{n_G}} D_{i,a}^{(q)}(g) |q, i, a\rangle \tag{2.7}$$

for  $g \in G$ . Here  $q$  uniquely labels an irreducible representation

$$D_{k,j}^{(q)}(g) = \langle q, k, a | \hat{T}_g | q, j, a \rangle \tag{2.8}$$

of  $G$  whose dimension is  $n_q$ ,  $a$  indexes different copies of each irreducible representation,  $|q, i, a\rangle$  are the basis states for which  $\hat{T}_g$  is block diagonal, the number of copies of each representation equals the dimension of the representation  $n_q$ , the sum over  $q$  ranges over all irreducible representations and the dimension  $D$  of the Hilbert space  $H$  spanned by the basis states  $|q, i, a\rangle$  is such that  $D = n_G$ . The reference states  $|g\rangle$  have the property that their “orientation” is changed by the action of the group, viz.  $\hat{T}(g')|g\rangle = |g' \circ g\rangle$  and  $\langle g|g'\rangle = \delta_{g,g'}$ , and so the set of reference states  $\{|g\rangle : g \in G\}$  forms an orthonormal basis for  $H$ . They completely break the symmetry of  $G$  in the sense that the action of the group  $\hat{T}(g')|g\rangle$  for  $g' \in G$  generates a set of mutually orthogonal states. In other words the reference states  $|g\rangle$  give a distinct orientation for each element of the group. We shall refer to systems of this kind as capable of completely breaking the symmetry of  $G$ .

We will also need to distinguish between different kinds of operators based on their effect on the symmetry or asymmetry. We already have one set representing the group  $G$  and can use it to define another. A unitary operator  $\hat{U}$  is called  $G$ -invariant if

$$\hat{U} \hat{T}_g \hat{\rho} \hat{T}_g^\dagger \hat{U}^\dagger = \hat{T}_g \hat{U} \hat{\rho} \hat{U}^\dagger \hat{T}_g^\dagger \tag{2.9}$$

for all  $g \in G$  and all states  $\hat{\rho}$ . It has been shown that  $G$ -invariant operations cannot increase the asymmetry of a state either individually or on average (Vaccaro *et al.* 2008).

We want to know how much information about the symmetry and asymmetry of a quantum system with respect to  $G$  is associated with the knowledge of its

density operator  $\hat{\rho}$ . For this we need to see how much information can be encoded using the symmetry and asymmetry of the system.

### 3. Information capacity of symmetry and asymmetry

#### (a) Asymmetry

We first examine how any asymmetry of  $\hat{\rho}$  can be used to send information. A state which is asymmetric is transformed by the actions of the group  $G = \{g\}$  to a different state. This means that the transformation

$$\hat{\rho} \mapsto \hat{\rho}_g = \hat{T}_g \hat{\rho} \hat{T}_g^\dagger \tag{3.1}$$

for  $g \in G$  can carry information. Moreover, it is straightforward to show that  $\mathcal{G}[\hat{T}_g \hat{\rho} \hat{T}_g^\dagger] = \mathcal{G}[\hat{\rho}]$  and so

$$W_G(\hat{T}_g \hat{\rho} \hat{T}_g^\dagger) = W_G\left(\sum_{g \in G} p_g \hat{T}_g \hat{\rho} \hat{T}_g^\dagger\right) = W_G(\hat{\rho}) \tag{3.2}$$

for all  $g \in G$ , where  $\{p_g\}$  is a probability distribution. This means that the operators  $\hat{T}_g$  do not change the value of the symmetry  $W_G(\hat{\rho})$  either individually or on average. Their use will ensure that the encoding involves only the asymmetry of the system.

We imagine an information theoretic scenario where one party, Alice, sends another, Bob, information encoded in the transformed states  $\hat{\rho}_g$ . Specifically Alice prepares the system in the transformed state  $\hat{\rho}_g$  for  $g \in G$  with probability  $p_g$  and sends it to Bob. We would like Alice to encode the maximum amount of information in the asymmetry of  $\hat{\rho}$ . If the probabilities  $p_g$  were not all equal then the averaged state prepared by Alice  $\hat{\rho}_{av} = \sum_g p_g \hat{T}_g \hat{\rho} \hat{T}_g^\dagger$  might well be asymmetric according to equation (2.4) in that

$$\hat{T}_g \hat{\rho}_{av} \hat{T}_g^\dagger \neq \hat{\rho}_{av} \tag{3.3}$$

for some  $g \in G$ . In this case the maximum amount of information is not guaranteed to be encoded because any asymmetry of  $\hat{\rho}_{av}$  could be used to encode additional information using the  $T_g$  operators. In contrast, if the probabilities are all equal the averaged prepared state is given by  $\hat{\rho}_{av} = \mathcal{G}[\hat{\rho}]$ , as defined in equation (2.2), which is symmetric because it is invariant to the actions of the group, i.e.

$$\hat{T}_g \left(\mathcal{G}[\hat{\rho}]\right) \hat{T}_g^\dagger = \mathcal{G}[\hat{\rho}] \text{ for } g \in G . \tag{3.4}$$

No further information can be encoded in  $\mathcal{G}[\hat{\rho}]$  using the  $T_g$  operators and so the encoding is maximal. We therefore stipulate that Alice prepares each state  $\hat{\rho}_g$  with equal probability,  $p_g = 1/n_G$  so that the averaged prepared state is  $\mathcal{G}[\hat{\rho}]$ .

On receiving the system, Bob makes a measurement to estimate the value of the parameter  $g$ . In particular, let Bob make the measurement  $\mathcal{M}$  described by the Kraus operators  $\hat{M}_k$  satisfying  $\sum_k \hat{M}_k^\dagger \hat{M}_k = \hat{\mathbf{1}}$  (Kraus 1983). Bob obtains result  $k$  with probability  $P_{k,g} = \text{Tr}(\hat{M}_k^\dagger \hat{M}_k \hat{\rho}_g)$  for the prepared state  $\hat{\rho}_g$ . The mutual information shared by Alice and Bob about their respective indices  $g$  and  $k$  is given by (Schumacher *et al.* 1996)

$$I(\mathcal{M} : \mathcal{G}) = H(\{p_g\}) - \sum_k P(k)H(\{P(g|k)\}) \tag{3.5}$$

where  $H(\{q_n\}) = -\sum_n q_n \log(q_n)$  is the Shannon entropy of the probability distribution  $\{q_n\}$ ,  $P(k) = \sum_g p_g P(k|g)$  is the average probability of getting result  $k$  and  $P(g|k) = p_g P(k|g)/P(k)$  is the conditional probability that Alice prepared the state  $\hat{\rho}_g$  given that Bob obtained measurement outcome  $k$ .

We define the information capacity,  $I_{\text{asym}}(\hat{\rho})$ , of the asymmetry of  $\hat{\rho}$  as the accessible information about the value of  $g$  carried by the ensemble of states  $\{\hat{\rho}_g : g = 1, 2, \dots, n_G\}$  sent to Bob. This is given by the maximum of  $I(\mathcal{M} : \mathcal{G})$  over all possible measurements  $\mathcal{M}$  by Bob,

$$I_{\text{asym}}(\hat{\rho}) = \max_{\mathcal{M}} I(\mathcal{M} : \mathcal{G}) , \tag{3.6}$$

and it is bounded above by Holevo’s theorem (Schumacher *et al.* 1996, Ruskai 2002):

$$I_{\text{asym}}(\hat{\rho}) \leq S(\mathcal{G}[\hat{\rho}]) - \frac{1}{n_G} \sum_{g \in G} S(\hat{\rho}_g) . \tag{3.7}$$

As  $\hat{T}_g$  is unitary,  $S(\hat{\rho}_g) = S(\hat{T}_g \hat{\rho} \hat{T}_g^\dagger) = S(\hat{\rho})$  for all  $g \in G$ , and we find equation (3.7) becomes (Vaccaro 2006, see also Gour *et al.* 2009)

$$I_{\text{asym}}(\hat{\rho}) \leq S(\mathcal{G}[\hat{\rho}]) - S(\hat{\rho}) . \tag{3.8}$$

Comparing the right side with equation (2.6) shows that the information capacity  $I_{\text{asym}}(\hat{\rho})$  is bounded by the entropic measure of the asymmetry  $A_G(\hat{\rho})$ .

There is another way of interpreting this result. The value of  $g$  indexes different “orientations” of the system due to its asymmetry. Information about  $g$  is therefore information about the asymmetry of system. There is a total of  $I_{\text{asym}}(\hat{\rho})$  bits of classical information about the asymmetry being sent to Bob and this is the maximum amount possible. We conclude that *Alice’s knowledge of the original state  $\hat{\rho}$  represents  $I_{\text{asym}}(\hat{\rho})$  bits of classical information about the asymmetry.*

(b) Symmetry

We now consider the analogous information theoretic scenario that uses the symmetry part of the state  $\hat{\rho}$ . To avoid using the asymmetry, we want the set of operations used for the encoding to leave the value of the asymmetry  $A_G(\hat{\rho})$  unchanged both on individual application and on average. This requirement is the complement of equation (3.2). A  $G$ -invariant unitary operator  $\hat{U}$  defined in equation (2.9) has the properties that  $S(\mathcal{G}[\hat{U} \hat{\rho} \hat{U}^\dagger]) = S(\hat{U} \mathcal{G}[\hat{\rho}] \hat{U}^\dagger) = S(\mathcal{G}[\hat{\rho}])$  and  $S(\hat{U} \hat{\rho} \hat{U}^\dagger) = S(\hat{\rho})$ , and so it does not change the asymmetry of  $\hat{\rho}$ , i.e.  $A_G(\hat{U} \hat{\rho} \hat{U}^\dagger) = A_G(\hat{\rho})$ . Let Alice use a subset of  $G$ -invariant unitary operators  $U = \{\hat{U}_1, \hat{U}_2, \dots, \hat{U}_N\}$  to encode information in the system by preparing the state

$$\hat{\rho}_j = \hat{U}_j \hat{\rho} \hat{U}_j^\dagger \tag{3.9}$$

with probability  $q_j$ . The choice of the subset must not change the asymmetry of the state on average and so

$$A_G(\hat{\rho}_{\text{av}}) = A_G(\hat{\rho}) \tag{3.10}$$

where  $\hat{\rho}_{\text{av}} = \sum_j q_j \hat{\rho}_j$ . Equation (3.10) ensures that the encoding involves only the symmetry of the system. Moreover, we want Alice to encode the maximum amount of information in the symmetry. This means that the averaged state after the encoding should have no symmetry, i.e.  $W_G(\hat{\rho}_{\text{av}}) = 0$  which, from equation (2.5), implies

$$S(\mathcal{G}[\hat{\rho}_{\text{av}}]) = \log(D) . \tag{3.11}$$

The system is then sent to Bob who makes a measurement to estimate the value of the parameter  $j$ . Let Bob's measurement be described as before. The mutual information shared by Alice and Bob about their respective indices  $j$  and  $k$  is given by

$$I(\mathcal{M} : \mathcal{U}) = H(\{p_j\}) - \sum_k P(k)H(\{P(j|k)\}) \tag{3.12}$$

where  $\mathcal{U}$  represents the encoding,  $P(k) = \sum_j p_j P(k|j)$  is the average probability of getting result  $k$  and  $P(j|k) = p_j P(k|j)/P(k)$  is the probability that Alice prepared the state  $\hat{\rho}_j$  given that Bob obtained measurement outcome  $k$ . We define the information capacity,  $I_{\text{sym}}(\hat{\rho})$ , of the symmetry of  $\hat{\rho}$  as the accessible information about the value of  $j$  carried by the ensemble of states  $\{\hat{\rho}_j : j = 1, 2, \dots, N\}$  sent to Bob. This is given by the maximum of  $I(\mathcal{M} : \mathcal{U})$  over all possible measurements  $\mathcal{M}$  by Bob,

$$I_{\text{sym}}(\hat{\rho}) = \max_{\mathcal{M}} I(\mathcal{M} : \mathcal{U}) . \tag{3.13}$$

From Holevo's theorem we find

$$I_{\text{sym}}(\hat{\rho}) \leq S(\hat{\rho}_{\text{av}}) - \sum_j q_j S(\hat{\rho}_j) , \tag{3.14}$$

and as the encoding is unitary  $S(\hat{\rho}_j) = S(\hat{\rho})$  and so

$$I_{\text{sym}}(\hat{\rho}) \leq S(\hat{\rho}_{\text{av}}) - S(\hat{\rho}) . \tag{3.15}$$

Equation (3.10) can be written as

$$S(\mathcal{G}[\hat{\rho}_{\text{av}}]) - S(\hat{\rho}_{\text{av}}) = S(\mathcal{G}[\hat{\rho}]) - S(\hat{\rho}) , \tag{3.16}$$

which can be rearranged to

$$S(\mathcal{G}[\hat{\rho}_{\text{av}}]) - S(\mathcal{G}[\hat{\rho}]) = S(\hat{\rho}_{\text{av}}) - S(\hat{\rho}) , \tag{3.17}$$

and so equation (3.15) can be written as

$$I_{\text{sym}}(\hat{\rho}) \leq S(\mathcal{G}[\hat{\rho}_{\text{av}}]) - S(\mathcal{G}[\hat{\rho}]) . \tag{3.18}$$

Using equation (3.11) then gives

$$I_{\text{sym}}(\hat{\rho}) \leq \log D - S(\mathcal{G}[\hat{\rho}]) . \tag{3.19}$$

Comparing the right side with equation (2.5) shows that the information capacity  $I_{\text{sym}}(\hat{\rho})$  is bounded by the entropic measure of the symmetry  $W_G(\hat{\rho})$ .

Moreover, the index  $j$  associated with the state  $\hat{\rho}_j$  that is sent to Bob represents information about the symmetry of the state  $\hat{\rho}$ . There is a total of  $I_{\text{sym}}(\hat{\rho})$  bits of classical information about the symmetry being sent to Bob and this is the maximum amount possible. This implies that *Alice's knowledge of the original state  $\hat{\rho}$  represents  $I_{\text{sym}}(\hat{\rho})$  bits of classical information about the symmetry.*

### 4. Duality of symmetry and asymmetry

#### (a) Duality relation

Combining the two expressions (3.8) and (3.19) yields the duality relation,

$$I_{\text{asym}}(\hat{\rho}) + I_{\text{sym}}(\hat{\rho}) \leq \ln(D) - S(\hat{\rho}), \tag{4.1}$$

which is the central result of this paper. It shows that the sum of the information capacities of the symmetry and asymmetry is bounded by the maximum classical information that can be carried by the system.

We could have anticipated this relation from the manner in which we designed the encodings: the encoding of  $I_{\text{asym}}(\hat{\rho})$  bits of information in the asymmetry of the state leaves the symmetry unchanged, and so a further  $I_{\text{sym}}(\hat{\rho})$  bits of information can subsequently be encoded in the symmetry of the same system (and vice versa for the reverse order of encodings). Indeed, the sets of operators,  $\{\hat{T}_g\}$  and  $\{\hat{U}_j\}$ , used in the two encodings commute in the sense of equation (2.9). The sum  $I_{\text{asym}}(\hat{\rho}) + I_{\text{sym}}(\hat{\rho})$ , therefore, is *necessarily* bounded by the maximum amount of classical information that can be carried by the system.

But the importance of the inequality (4.1) does not lie in the actual value of the bound. Rather, it is the fact (4.1) is a duality between information about the symmetry and the asymmetry of the system. Knowing that the state of the system is  $\hat{\rho}$  gives us  $I_{\text{asym}}(\hat{\rho})$  bits of information about its asymmetry and  $I_{\text{sym}}(\hat{\rho})$  bits of information about its symmetry. Regardless of how the system is prepared, we can never have both  $\log(D)$  bits of information about its symmetry and  $\log(D)$  bits of information about its asymmetry. Inequality (4.1) therefore *represents a duality between our knowledge of the asymmetry and the symmetry of the system.*

In §2 we expressed classical particle and wave character of a quantum system in terms the asymmetry and symmetry, respectively, with respect to a group. Inequality (4.1) *represents a duality between our knowledge of the particle and the wave character of the system* in the same manner.

#### (b) Examples

It is instructive to explore the duality relation (4.1) for states of particular interest. Consider a system which is capable of completely breaking the symmetry of  $G$  as discussed in §2(c). Let the system be prepared in one of the reference states defined in equation (2.7), say  $\hat{\rho} = |g\rangle\langle g|$  where  $g \in G$ . In this case the possible density operators prepared by Alice are simply  $\hat{T}_{g'}\hat{\rho}\hat{T}_{g'}^\dagger = |g' \circ g\rangle\langle g' \circ g|$  for  $g' \in G$ ; these density operators are mutually commuting and so the equality in Holevo's bound in equation (3.8) is satisfied (Ruskai 2002). Moreover  $S(\hat{\rho}) = 0$  and the averaged state sent to Bob is proportional to the identity operator which means  $S(\mathcal{G}[\hat{\rho}]) = \log(D)$ . The reference states therefore exhibit the maximum information capacity possible, i.e.

$$I_{\text{asym}}(|g\rangle\langle g|) = \log(D) \tag{4.2}$$

and so the bound in equation (3.8) is achievable. As  $S(\mathcal{G}[\hat{\rho}]) = \log(D)$  we find from equation (3.19) that

$$I_{\text{sym}}(|g\rangle\langle g|) = 0 \tag{4.3}$$

and so the symmetry has zero information capacity. Combining these results shows that the equality in equation (4.1) is satisfied:

$$I_{\text{asym}}(|g\rangle\langle g|) + I_{\text{sym}}(|g\rangle\langle g|) = \log(D) . \tag{4.4}$$

Another way to state these results is that preparing the system in a reference state gives maximum information about the asymmetry and minimum information about the symmetry of the system.

There are no symmetric pure states for irreducible representations of dimension greater than 1. In particular, a uniform linear superposition of all the reference states of the form  $|\phi\rangle \propto \sum_{g \in G} |g\rangle$  satisfies the symmetric condition  $\mathcal{G}[|\phi\rangle\langle\phi|] = |\phi\rangle\langle\phi|$ , however, this state is easily shown to be the trivial representation of dimension 1. For this state we find

$$I_{\text{sym}}(|\phi\rangle\langle\phi|) = \log(D) \tag{4.5}$$

for an encoding in which Alice sends mutually commuting density operators  $\hat{U}_j|\phi\rangle\langle\phi|\hat{U}_j^\dagger$  (Ruskai 2002). As it is a symmetric pure state,  $S(\mathcal{G}[|\phi\rangle\langle\phi|]) = 0$  and so from equation (3.8)

$$I_{\text{asym}}(|\phi\rangle\langle\phi|) = 0 . \tag{4.6}$$

Hence

$$I_{\text{asym}}(|\phi\rangle\langle\phi|) + I_{\text{sym}}(|\phi\rangle\langle\phi|) = \log(D) \tag{4.7}$$

and so the equality in equation (4.1) is satisfied. Preparing the system in a pure symmetric state clearly gives maximum information about the symmetry and no information about the asymmetry of the system.

### 5. Composite systems

#### (a) Duality relation for composite system

It is rather straightforward to extend the analysis above to a composite system consisting of two identical systems, labelled  $a$  and  $b$ . Let each system be of the kind we have been considering, with a  $D$ -dimensional Hilbert space  $H$  that carries a representation of the group  $G$ . The Hilbert space  $H \otimes H$  of the composite  $ab$ -system carries a representation of  $G$  which acts *globally* (or collectively) in the sense that its unitary representation  $\tau^{(ab)} = \{\hat{T}_g^{(ab)} : g \in G\}$ , where

$$\hat{T}_g^{(ab)} \equiv \hat{T}_g \otimes \hat{T}_g , \tag{5.1}$$

applies the same group action to each system (Fulton & Harris 1991). Here  $\hat{A} \otimes \hat{B}$  represents a tensor product of the operators  $\hat{A}$  and  $\hat{B}$  which act on the  $a$  and  $b$  systems, respectively.

It is straightforward to show that the analysis of the previous section with respect to the group representation  $\tau^{(ab)}$  on  $H \otimes H$  gives the bound on the information capacity  $I_{\text{asym}}^{(ab)}$  of the composite system for an encoding in terms of the operators  $\hat{T}_g^{(ab)}$  as

$$I_{\text{asym}}^{(ab)}(\hat{\rho}) \leq S(\mathcal{G}^{(ab)}[\hat{\rho}]) - S(\hat{\rho}) \tag{5.2}$$

for the state  $\hat{\rho}$  of the  $ab$ -system with

$$\mathcal{G}^{(ab)}[\hat{\rho}] \equiv \frac{1}{|G|} \sum_{g \in G} \hat{T}_g^{(ab)} \hat{\rho} \hat{T}_g^{(ab)\dagger} . \quad (5.3)$$

Similarly, the bound on the information capacity of an encoding in terms of the symmetry of  $\hat{\rho}$  is

$$I_{\text{sym}}^{(ab)}(\hat{\rho}) \leq \log(D^2) - S(\mathcal{G}^{(ab)}[\hat{\rho}]) . \quad (5.4)$$

Hence the duality relation for the  $ab$ -system is

$$I_{\text{asym}}^{(ab)}(\hat{\rho}) + I_{\text{sym}}^{(ab)}(\hat{\rho}) \leq 2 \log(D) - S(\hat{\rho}) . \quad (5.5)$$

The  $ab$ -system has the capacity to carry twice the classical information as each single system and this is reflected in the larger bound here compared to equation (4.1).

(b) Examples

It is interesting to compare the bounds on the symmetry and asymmetry for a composite of two systems each of which is capable of completely breaking the symmetry of a finite group  $G$ . The product reference state  $|g\rangle \otimes |g\rangle$  gives the maximum information capacity of the asymmetry of the  $ab$ -system of

$$I_{\text{asym}}^{(ab)}(|g\rangle \otimes |g\rangle) = \log(D) . \quad (5.6)$$

Here, and in the following, we write the state as a simple Dirac ket rather than a density operator to make the notation easier. It might have been expected that the asymmetry would be twice this amount because the sum of the asymmetry for each system in equation (4.2) is  $2 \log(D)$ . However,  $I_{\text{asym}}^{(ab)}$  is the global asymmetry of the  $ab$ -system for which  $G$  acts identically on the individual systems according to equation (5.1). The information capacity of the symmetry is easily found to be

$$I_{\text{sym}}^{(ab)}(|g\rangle \otimes |g\rangle) = \log(D) \quad (5.7)$$

and so the product reference state is as asymmetric as it is symmetric. Conversely, the state  $|\varphi\rangle = \sum_{g \in G} |g\rangle \otimes |g\rangle / \sqrt{nG}$  has the minimum asymmetry and maximum symmetry:

$$I_{\text{asym}}^{(ab)}(|\varphi\rangle) = 0, \quad I_{\text{sym}}^{(ab)}(|\varphi\rangle) = 2 \log(D) . \quad (5.8)$$

Although the state  $|\varphi\rangle$  has no (global) asymmetry, nevertheless it does possess *local* asymmetry in the sense that it is transformed by the action of  $\hat{T}_g \otimes \hat{T}_h$  for particular choices of  $g, h \in G$  to a different state. Indeed the transformed state

$$\hat{T}_g \otimes \hat{T}_h |\varphi\rangle = \hat{T}_e \otimes \hat{T}_r |\varphi\rangle \quad (5.9)$$

is orthogonal to  $|\varphi\rangle$  unless  $r = e$ , where  $r = h \circ g^{-1}$  and  $e$  is the identity element. Thus while the composite system as a whole is symmetric, it has asymmetric parts internally. Moreover, the transformed state in equation (5.9) is also globally symmetric. Evidently the asymmetry of one component system is compensated by the asymmetry of the other. The state  $|\varphi\rangle$  is maximally entangled and its symmetry is due to it belonging to a 1 dimensional representation of  $G$  on  $H \otimes H$ . This situation is related to the superdense coding scheme of Bennett & Wiesner (1992) which we examine in the next section.

These results are easily extended to the general case of  $n$  systems as follows:

$$I_{\text{asym}}^{(ab\dots)}(|g\rangle^{\otimes n}) = \log(D) , \quad I_{\text{sym}}^{(ab\dots)}(|g\rangle^{\otimes n}) = (n - 1) \log(D) \quad (5.10)$$

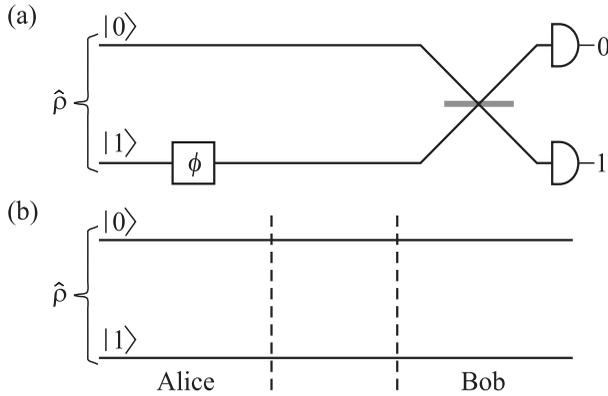


Figure 2. Quantum circuit diagrams of (a) a two path interferometer and (b) its representation as a communication channel. A wavicle enters from the left in a given state  $\hat{\rho}$ . In (a) interference with respect to the phase shift  $\phi$  is observed in the outputs of the detectors on the right. In (b) Alice can encode information in the symmetry or asymmetry of  $\hat{\rho}$ . The maximum amount of information that can be transmitted to Bob in this way is the information Alice has about the symmetry or asymmetry from her knowledge of  $\hat{\rho}$ .

whereas

$$I_{\text{asym}}^{(ab\dots)}\left(\frac{1}{\sqrt{n_G}} \sum_g |g\rangle^{\otimes n}\right) = 0, \quad I_{\text{sym}}^{(ab\dots)}\left(\frac{1}{\sqrt{n_G}} \sum_g |g\rangle^{\otimes n}\right) = n \log(D). \quad (5.11)$$

Clearly more systems implies greater symmetry and thus more wavelike character. These results stem directly from the manner in which the group acts globally, according to equation (5.1). For the cases considered here, that is for finite groups and systems that can completely break the symmetry according to equation (2.7), the maximum of  $I_{\text{asym}}^{(ab\dots)}$  remains fixed at  $\log(D) = \log(n_G)$  as the number  $n$  of systems grows. The total amount of classical information that can be carried by the composite system is  $n \log(D)$ , and so the maximum amount of information that can be carried by the symmetry also scales linearly with  $n$ .

### 6. Applications

#### (a) Two-path interferometer

As an application of the single system case let's examine a two-path interferometer. A quantum system, or "wavicle" if you will (Eddington 1928), enters the left side of the interferometer illustrated in figure 2(a) in the state  $\hat{\rho}$ , passes through the beam splitter and is detected by two detectors on the right. Let the states representing the wavicle occupying the upper and lower paths of the interferometer be  $|0\rangle$  and  $|1\rangle$ , respectively. A phase shift  $\phi$  is applied to the lower path as shown. The action of the phase shifter and beam splitter on the state of the

wavicle is described by  $\hat{\rho} \mapsto \hat{U}(\phi)\hat{\rho}\hat{U}^\dagger(\phi)$  where  $\hat{U}(\phi) = \frac{1}{\sqrt{2}}(\hat{\mathbf{1}} + i\hat{\sigma}_y) \exp(i\phi|1\rangle\langle 1|)$  and  $\hat{\sigma}_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|)$ . The probability that the upper detector detects the wavicle depends on  $\phi$  and represents an interference pattern whose visibility is given by  $V = 2|\langle 1|\hat{\rho}|0\rangle|$ .

The symmetry-asymmetry duality arises when we treat the wavicle–two-path system as a communication channel between two parties, Alice and Bob, as illustrated in figure 2(b). The interchange of the two paths of the communication channel is the two-state equivalent of the continuous spatial translation discussed in §2(a). Just as we defined the symmetry of a classical wave in §2(a) in terms of the invariance to spatial translations, here symmetry is defined in terms of the invariance to path interchange. The symmetry group associated with this problem is therefore given by  $G = \{e, x\}$  where  $e$  is the identity element and  $x$  represents the interchange of the paths.

A suitable unitary representation of  $G$  is given by  $\tau = \{\hat{\mathbf{1}}, \hat{\sigma}_x\}$  where  $\hat{\mathbf{1}}$  is the identity operator and  $\hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$ . Alice can encode information in the symmetry or asymmetry of  $\hat{\rho}$  with respect to  $G$  and transmit this information to Bob via the wavicle. In particular, the information capacity of the asymmetry,  $I_{\text{asym}}(\hat{\rho})$ , represents classical information about the asymmetry that Alice has from her knowledge that the state is  $\hat{\rho}$ . This information represents what she knows about the behaviour of the wavicle under the action of the operator  $\hat{\sigma}_x$  which interchanges the paths. In other words, it represents what she knows about the distinguishability of the paths from her knowledge of  $\hat{\rho}$  and so  $I_{\text{asym}}(\hat{\rho})$  plays essentially the same role as Englert’s which-way information (Englert 1996).

Consider first the states  $|n\rangle$  where  $n = 0$  or  $1$ . These represent the wavicle occupying one of the paths and so they are particle-like states. From equations (3.8), (3.19) and (4.1) we find

$$I_{\text{asym}}(|n\rangle) = 1, \quad I_{\text{sym}}(|n\rangle) = 0, \quad I_{\text{asym}}(|n\rangle) + I_{\text{sym}}(|n\rangle) = \log(D) \quad (6.1)$$

which shows that the state is maximally asymmetric or particle like. The same states would give a visibility  $V$  of zero in the interferometer in figure 2(a).

Next consider the states representing the wavicle occupying an equal superpositions of the two paths,  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ . In this case we find

$$I_{\text{asym}}(|\pm\rangle) = 0, \quad I_{\text{sym}}(|\pm\rangle) = 1, \quad I_{\text{asym}}(|\pm\rangle) + I_{\text{sym}}(|\pm\rangle) = \log(D) \quad (6.2)$$

which shows that the state is maximally symmetric or wave like. The same states would give a visibility  $V$  of unity in the interferometer. Evidently the symmetry of the system with respect to the group  $G$  encompasses the classical wave-like properties that are probed by the interferometer.

We argued in §2(b) that interferometry can be viewed as a particular communication protocol that encodes information in the symmetry of a system. Let’s now examine this idea in the context of the two-path interferometer in figure 2(a). Imagine that Alice uses the identity and the phase shift operators to encode information using the initial state  $|+\rangle$  and Bob uses the beam splitter and detector arrangement for decoding. We saw in §3(b) that for the encoding to use only the symmetry of the system it must not change the asymmetry. This is clearly the case as the initial state  $|+\rangle$  and the possible encoded states  $|\pm\rangle$  all have zero

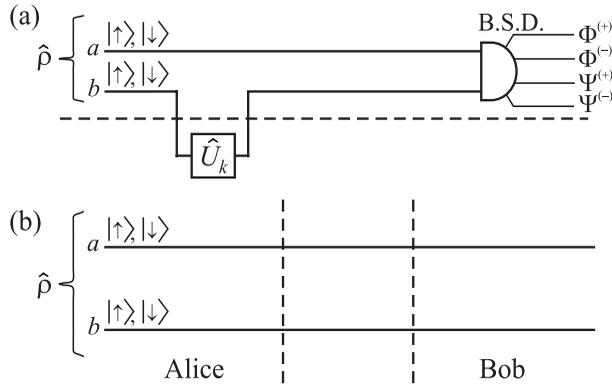


Figure 3. Quantum circuit diagrams of (a) the superdense coding scheme and (b) its representation as a symmetric and asymmetric encoding communication channel. A pair of spin- $\frac{1}{2}$  particles labelled  $a$  and  $b$  enters from the left in a given state  $\hat{\rho}$ . In (a) a local operator  $\hat{U}_k \in \{\hat{\mathbf{1}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$  is applied to the lower spin and then both spins pass to the Bell state discriminator (B.S.D.). In (b) Alice can encode information using the symmetry or asymmetry of  $\hat{\rho}$  and send it to Bob.

asymmetry as does the average encoded state. But we should also check that the encoding operators are  $G$ -invariant in the sense of equation (2.9). For this, first note that the Pauli operators satisfy

$$\hat{\sigma}_i \hat{\sigma}_j = i e_{ijk} \hat{\sigma}_k \quad (\text{for } i, j \text{ and } k \text{ all different}) \quad (6.3)$$

where  $e_{ijk}$  is the permutation symbol and  $\hat{\sigma}_1 = \hat{\sigma}_x$ ,  $\hat{\sigma}_2 = \hat{\sigma}_y$  and  $\hat{\sigma}_3 = \hat{\sigma}_z$ . It immediately follows that

$$\hat{\sigma}_x (\hat{\sigma}_z)^n \hat{\rho} (\hat{\sigma}_z)^n \hat{\sigma}_x = (\hat{\sigma}_z)^n \hat{\sigma}_x \hat{\rho} \hat{\sigma}_x (\hat{\sigma}_z)^n \quad (6.4)$$

and so

$$\hat{\sigma}_x f(\hat{\sigma}_z) \hat{\rho} f(\hat{\sigma}_z) \hat{\sigma}_x = f(\hat{\sigma}_z) \hat{\sigma}_x \hat{\rho} \hat{\sigma}_x f(\hat{\sigma}_z) \quad (6.5)$$

for any function  $f(\cdot)$  with a power series expansion. This means that the phase shift operator  $\exp(i\phi|1\rangle\langle 1|) = \exp[i\phi(\hat{\mathbf{1}} - \hat{\sigma}_z)]$  is indeed  $G$ -invariant. As the identity operator is trivially  $G$ -invariant, we conclude that the two-path interferometer can act as a communication channel that encodes information in the symmetry only, as anticipated.

### (b) Superdense coding

The superdense coding scheme of Bennett & Wiesner (1992) allows 2 bits of classical information to be communicated using a single spin- $\frac{1}{2}$  particle and pre-existing entanglement. Figure 3(a) illustrates the scheme. A pair of spin- $\frac{1}{2}$  particles labelled  $a$  and  $b$  are prepared in the Bell state  $|\Psi^{(-)}\rangle$ . Here the four Bell states are given by  $|\Phi^{(\pm)}\rangle = (|\uparrow\rangle|\uparrow\rangle \pm |\downarrow\rangle|\downarrow\rangle)/\sqrt{2}$  and  $|\Psi^{(\pm)}\rangle = (|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$  where  $|\uparrow\rangle$  and  $|\downarrow\rangle$  represent eigenstates of the  $z$  component of spin with eigenvalues  $\hbar/2$  and  $-\hbar/2$ , respectively (Braunstein *et al.* 1992). A local operation  $\hat{U}_k \in \{\hat{\mathbf{1}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ ,

where  $\hat{\sigma}_x$ ,  $\hat{\sigma}_y$  and  $\hat{\sigma}_z$  are Pauli spin operators, applied to the lower spin transforms the state of the pair into one of the Bell states. The value of  $k$  can be determined by a detector which discriminates between these four states.

Although we have used the Bell state  $|\Psi^{(-)}\rangle$  in this example, the same coding and detection scheme in figure 3(a) works with the system initially prepared in any of the four Bell states. Superdense coding can also work with any maximally entangled pure state, but in that case the coding and detection methods in figure 3(a) need to be modified accordingly. For brevity, we shall only treat the encoding and detection scheme depicted explicitly in figure 3(a).

We now reexamine the scheme using the information capacities of the symmetry and asymmetry for a composite system. We first need to identify the symmetry group associated with the problem. The symmetry operations will be analogous to the spatial translations mentioned in §2(a) and the path exchange for the previous example. Specifically, we need to look for those operations that leave the initial prepared state unchanged in the sense of equation (2.1), which for the composite system here is

$$\mathcal{G}^{(ab)}[\hat{\rho}] = \hat{\rho} \ , \tag{6.6}$$

where  $\mathcal{G}^{(ab)}[\cdot]$  is given by equations (5.1) and (5.3). One might be tempted to say that the operations we want are the group of global rotations  $\{\hat{T}_g \otimes \hat{T}_g : \hat{T}_g \in \text{SU}(2)\}$  because these operations leave the singlet state  $|\Psi^{(-)}\rangle$  invariant. However, any of the four Bell states can be used in the same experimental arrangement in figure 3(a) whereas only the singlet state is invariant under this group. The symmetry described by this group is, therefore, not sufficiently general. Instead, it is straightforward to show that a subgroup comprising the global Pauli operators, i.e.

$$G = \{\hat{\mathbf{1}} \otimes \hat{\mathbf{1}}, \hat{\sigma}_x \otimes \hat{\sigma}_x, \hat{\sigma}_y \otimes \hat{\sigma}_y, \hat{\sigma}_z \otimes \hat{\sigma}_z\} \ , \tag{6.7}$$

leaves all four Bell states invariant in the sense of equation (6.6). We therefore take this group to define the symmetry of the problem.

The maximum value of asymmetry for pure states occurs when  $\mathcal{G}^{(ab)}[\hat{\rho}]$  is proportional to the identity operator. An example of a state with this property is  $|\phi\rangle = (|\uparrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle)/\sqrt{2} = (|\Phi^{(+)}\rangle + |\Phi^{(-)}\rangle + |\Psi^{(+)}\rangle + |\Psi^{(-)}\rangle)/2$  for which we find, using equations (5.2), (5.4) and (5.5), that

$$I_{\text{asym}}^{(ab)}(|\phi\rangle) = 2 \ , \quad I_{\text{sym}}^{(ab)}(|\phi\rangle) = 0 \ , \quad I_{\text{asym}}^{(ab)}(|\phi\rangle) + I_{\text{sym}}^{(ab)}(|\phi\rangle) = 2 \log(D) \tag{6.8}$$

where  $D = 2$  is the dimension of the Hilbert space of each spin. The bound of the duality equation (5.5) is reached for zero information about the symmetry, and so this state is maximally asymmetric.

In contrast, all Bell states  $|B\rangle \in \{|\Phi^{(+)}\rangle, |\Phi^{(-)}\rangle, |\Psi^{(+)}\rangle, |\Psi^{(-)}\rangle\}$  are found to satisfy

$$I_{\text{asym}}^{(ab)}(|B\rangle) = 0 \ , \quad I_{\text{sym}}^{(ab)}(|B\rangle) = 2 \ , \quad I_{\text{asym}}^{(ab)}(|B\rangle) + I_{\text{sym}}^{(ab)}(|B\rangle) = 2 \log(D) \tag{6.9}$$

and so they have maximal symmetry. In particular, 2 bits of classical information can be encoded in their symmetry.

It remains for us to show that the superdense coding scheme is described by the formalism of §5. The operators used in the superdense coding scheme are given by

$$U = \{\hat{\mathbf{1}} \otimes \hat{\mathbf{1}}, \hat{\mathbf{1}} \otimes \hat{\sigma}_x, \hat{\mathbf{1}} \otimes \hat{\sigma}_y, \hat{\mathbf{1}} \otimes \hat{\sigma}_z\} \quad (6.10)$$

and it is straightforward to show using equation (6.3) that they are  $G$ -invariant with respect to  $G$  defined in equation (6.7). The initial state of the scheme can be any of the bell states  $|B\rangle$  and these have no asymmetry and neither do any of the possible encoded states. The superdense coding scheme therefore encodes the information in the symmetry of the spin particles in accord with §5.

Comparing this with the previous example shows that one can view superdense coding as being essentially an *interference phenomena* involving wave-like properties with respect to the group  $G$  in equation (6.7). In this view the different outputs at the Bell-state discriminator correspond to interference fringes.

Although all Bell states  $|B\rangle$  are globally symmetric with respect to  $G$  they are asymmetric in the local sense. In fact the operators  $\hat{U}_k \in U$  used in the superdense coding scheme can be written as  $\hat{T}_e \otimes \hat{T}_g$  with  $e$  being the identity element and  $\hat{T}_g \in \{\hat{\mathbf{1}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ . This encoding is in the form of equation (5.9). Thus the 2 bits of classical information is encoded in the *local* asymmetry of  $|B\rangle$ .

## 7. Discussion

We began with the observation that a classical point particle breaks translational symmetry whereas a classical wave of uniform amplitude does not. A quantum system can be in a superposition of many positions which gives the system varying degrees of translational symmetry. The superposition state also allows the system to produce interference patterns in an interferometer and thus to exhibit wave-like character. The more translational symmetry the system has, the more wave like it is. Conversely, the less translational symmetry the system has, the more particle like it is. The broad aim of this paper is to elevate these associations to formal definitions of particle and wave nature for arbitrary quantum systems and arbitrary symmetry groups. To do this we quantified the symmetry and asymmetry of a system with respect to an arbitrary group  $G$  in terms of the amount of information each can carry. This led to the particle-wave duality relations for a single system (4.1) and for a composite system (5.5).

We then used the duality relations to recast the well-known concepts of two-path interferometry and superdense coding in new light. In both cases the first task in applying the duality relations was to find the symmetry group associated with the initial states of the two schemes. This entailed finding operations which were analogous to the spatial translations that leave a classical wave invariant. The operations found were path exchange for the two-path interferometer and global Pauli operators for the superdense coding scheme. We showed how the two-path interferometer can be viewed as a communication channel which encodes information in the symmetry of the initial state. Conversely, we showed how the superdense coding scheme can be viewed as an interferometer whose operation is to use the wave-like symmetry of its initial state to produce outputs that correspond to interference fringes.

The outcomes of this study are fourfold. First, it shows how a communication channel based on symmetry can be viewed as an interferometer, and vice versa. As an interferometer demonstrates the degree to which a system has wave-like properties, it shows how arbitrary symmetries can be interpreted in terms of wave-like and particle-like character. Second, it also establishes the duality relation that governs the amount of information that is available about each character from knowledge of the state  $\hat{\rho}$  of the system. Third, in doing so it replaces pairs of conjugate observables, such as position and momentum, with *sets of operators* associated with the classical notions of particle and wave. The set of operators belonging to the group  $G$  manipulate the particle-like asymmetry, and the set of operators that are  $G$ -invariant manipulate the wave-like symmetry of the system. Fourth, this approach avoids the highly mathematical nature of the mutually unbiased bases method by offering a way of studying particle-wave duality that is *general and yet retains a consistent physical interpretation*. These outcomes answer the questions posed in the Introduction.

Finally, we end with a connection between the duality studied here and complementarity. Although we have not made explicit reference to it, there must exist some prior measurement on the system in order to be able to know anything about its state. As pointed out by Bohr, measurement schemes that reveal unambiguous information about complementary observables are mutually exclusive (1935). Our knowledge of  $\hat{\rho}$  is therefore subject to the same limitations of complementarity. These limitations are implicit in the duality relations as limits on what can be known about the system in terms of its symmetry and asymmetry.

### Acknowledgment

I would like to thank Prof. Howard Wiseman for encouraging discussions. This research was conducted by the Australian Research Council Centre of Excellence for Quantum Computation and Communication Technology (Project number CE110001027).

### References

- Arthurs E. & Kelly, J.L. 1965 On the Simultaneous Measurement of a Pair of Conjugate Observables. *Bell. Sys. Tech. J.* **44**, 725–729.
- Barbieri, M., Goggin, M. E., Almeida, M. P., Lanyon, B. P. & White, A. G. 2009 Complementarity in variable strength quantum non-demolition measurements. *New J. Phys.* **11**, 093 012.
- Bartlett, S. D. & Wiseman, H. M. 2003 Entanglement constrained by superselection rules. *Phys. Rev. Lett.* **91**, 097 903.
- Bartlett, S.D., Rudolph, T. & Spekkens, R.W. 2007 Reference frames, superselection rules, and quantum information. *Rev. Mod. Phys.* **79**, 555–609 (DOI: 10.1103/RevModPhys.79.555).
- Bennett, C.H. & Wiesner, S.J. 1992 Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. *Phys. Rev. Lett.* **69**, 2881–2884.
- Berta, M., Christandl, M., Colbeck, R., Renes J. M. & Renner R. 2010 The uncertainty principle in the presence of quantum memory. *Nature Physics* **6**, 659–662.

- Björk, G. & Karlsson, A. 1998 Complementarity and quantum erasure in welcher Weg experiments. *Phys. Rev. A* **58**, 3477–3483.
- Bohr, N. 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **48**, 696.
- Braunstein, S.L. Mann, A. & Revzen, M. 1992 Maximal violation of Bell inequalities for mixed states. *Phys. Rev. Lett.* **68**, 3259–3261 (DOI: 10.1103/PhysRevLett.68.3259).
- Brouard, S. & Muga, J.G. 1998 Collisional Transitory Enhancement of the High Momentum Components of a Quantum Wave Packet. *Phys. Rev. Lett.* **81**, 2621–2625 (DOI: 10.1103/PhysRevLett.81.2621).
- Eddington, A. S. (1928) *The nature of the physical world*, p. 201. New York: Macmillan.
- Englert, B. -G. 1996 Fringe visibility and which-way information: an inequality. *Phys. Rev. Lett.* **77**, 2154–2157.
- Feynman, R. P., Leighton R. B. & Sands, M. 1963 *The Feynman lectures on physics*, vol. 3, ch. 1, p. 1. Reading, Massachusetts: Addison-Wesley Publishing.
- Fulton, W. & Harris, J. 1991 *Representation theory: a first course*. Berlin: Springer-Verlag.
- Gibbons, K. S., Hoffman, M. J. & Wootters, W. K. 2004 Discrete phase space based on finite fields. *Phys. Rev. A* **70**, 062 101.
- Gottesman, D. 1996 Class of quantum error-correcting codes saturating the quantum Hamming bound. *Phys. Rev. A* **54**, 1862–1868.
- Gour, G., Marvian, I. & Spekkens, R.W. 2009 Measuring the quality of a quantum reference frame: the relative entropy of frameness. *Phys. Rev. A* **80**, 012 307.
- Grangier, P., Roger G. & Aspect, A. 1986 Experimental evidence for a photon anticorrelation effect on a beam splitter: a new light on single-photon interferences. *Europhys. Lett.* **1**, 173–179.
- Kalev, A., Khanna, F. C. & Revzen, M. 2009 Partially unbiased entangled bases. *Phys. Rev. A* **80**, 022 112.
- Kitaev, A., Mayers D. & Preskill, J. 2004 Superselection rules and quantum protocols. *Phys. Rev. A* **69**, 052 326.
- Kraus, K. 1983 *States, Effects and Operations: Fundamental Notions of Quantum Theory*. Berlin: Springer-Verlag.
- Kurzynski, P., Kaszub, W. & Czechlewski, M. 2010 Mutually unbiased bases and complementary spin 1 observables. *J.Phys. A* **43**, 265 303.
- Luis, A. 2003 Complementarity and duality relations for finite-dimensional systems. *Phys. Rev. A* **67**, 032 108.
- Luis, A. 2004 Operational approach to complementarity and duality relations. *Phys. Rev. A* **70**, 062 107.
- Maassen, H. & Uffink, J. B. M. 1988 Generalized entropic uncertainty relations. *Phys. Rev. Lett.* **60**, 1103–1106.
- Miyadera, T. & Imai, H. 2006 Information-disturbance theorem for mutually unbiased observables. *Phys. Rev. A* **73**, 042 317.
- Nielsen, M. A. & Chuang, I. L. 2010 *Quantum Computation and Quantum Information*. New York: Cambridge University Press.
- Ozawa, M. 2004 Uncertainty relations for joint measurements of noncommuting observables. *Phys. Lett.* **320A**, 367–374.
- Pegg, D. T., Vaccaro, J. A. & Barnett, S. M. 1990 Quantum-optical phase and canonical conjugation. *J. Mod. Optics* **37**, 1703–1710.
- Pérez Prieto, A.L., Brouard, S. & Muga, J.G. 2001 Transient interference of transmission and incidence. *Phys. Rev. A* **64**, 012710 (DOI: 10.1103/PhysRevA.64.012710).
- Pitaevskii L. & Stringari S. 1999 Interference of Bose-Einstein Condensates in Momentum Space. *Phys. Rev. Lett.* **83**, 4237–4240 (DOI: 10.1103/PhysRevLett.83.4237).

- Rojas Gonzalez, A., Vaccaro, J.A. & Barnett, S.M. 1995 Entropic uncertainty relations for canonically conjugate operators. *Phys. Lett.* **205A**, 247–254.
- Rauch, H. 1993 Phase space coupling in interference and EPR experiments. *Phys. Lett. A* **173**, 240–242 (DOI: 10.1016/0375-9601(93)90270-A).
- Ruschhaupt, A. del Campo, A. & Muga J. G. 2009 Momentum-space interferometry with trapped ultracold atoms. *Phys. Rev. A* **79**, 023616 (DOI: 10.1103/PhysRevA.79.023616).
- Ruskai, M. B., 2002 Inequalities for quantum entropy: A review with conditions for equality. *J. Math. Phys.* **43**, 4358 (DOI: 10.1063/1.1497701); erratum 2005 *J. Math. Phys.* **46**, 019 901 (DOI: 10.1063/1.1824214).
- Schumacher, B., Westmoreland, M. & Wootters, W. K. 1996 Limitation on the amount of accessible information in a quantum channel. *Phys. Rev. Lett.* **76**, 3452–3455.
- Schwinger, J. 1960 Unitary Operator Bases. *P. Natl. Acad. Sci. USA* **46**, 570–579.
- Scully, M. O., Englert, B. -G. & Walther, H. 1991 Quantum optical tests of complementarity. *Nature* **351**, 111–116.
- Vaccaro, J. A. 2006 Group theoretic formulation of complementarity. In *Proceedings of the 8th International Conference on Quantum Communication, Measurement and Computing* (eds O. Hirota, J.H. Shapiro & M. Sasaki), pp. 421–424. Tokyo: NICT Press (Preprint available at arXiv:1012.3532v1).
- Vaccaro, J. A., Anselmi, F., Wiseman, H. M. & Jacobs, K. 2008 Tradeoff between extractable mechanical work, accessible entanglement, and ability to act as a reference system, under arbitrary superselection rules. *Phys. Rev. A* **77**, 032 114.
- Wiseman, H. M. 1998 Extending Heisenberg’s measurement-disturbance relation to the twin-slit case. *Found. Phys.* **28**, 1619–1631.
- Wootters, W. K. & Zurek, W. H. 1979 Complementarity in the double-slit experiment: Quantum nonseparability and a quantitative statement of Bohr’s principle. *Phys. Rev. D* **19**, 473–484.
- Wootters, W.K. 1987 A Wigner-function formulation of finite-state quantum mechanics. *Ann. Phys.* **176** 1–21.