

Two-Dimensional Finite-Difference Modeling of Media With Inclined Uniaxial Conductivity With an Equivalent Biaxial Conductivity Tensor for Homogeneous TM-Type Wave Propagation Problems

Glenn A. Wilson, *Student Member, IEEE*, and David V. Thiel, *Senior Member, IEEE*

Abstract—The principle of numerically modeling the surface impedance of a homogeneous transverse magnetic (TM)-type plane wave incident upon an inhomogeneous half-space with inclined uniaxial electrical anisotropy as an equivalent half-space with fundamental electrical biaxial anisotropy is demonstrated. The self-consistent impedance method is introduced and shown to accurately model the surface impedance response of these two-dimensional (2-D) induction problems at low frequencies relevant to surface impedance geophysics, though there is inaccuracy in the surface impedance phase as the frequency is increased. While the impedance method has been introduced to demonstrate this modeling concept, the modeling principles introduced can be applied to other 2-D numerical methods.

Index Terms—Electrical anisotropy, TM-type wave propagation, two-dimensional (2-D) finite-difference methods.

I. INTRODUCTION

THE DEVELOPMENT of analytical and numerical modeling techniques for the electromagnetic fields in electrically anisotropic and inhomogeneous media is of geophysical interest because such media approximates the macroscopic conductivity properties of sedimentary rocks, where the conductivity normal to the bedding plane is usually lower than the conductivity parallel to the bedding plane [1]. As discussed by Wilson and Thiel [2], one-dimensional analytical solutions for the surface impedance of a homogeneous plane wave incident upon a half-space with inclined uniaxial anisotropy, where the conductivity tensor is rotated about one of the horizontal axes, have been developed over the past 40 years. However, with the exception of the uniaxial anisotropy solution of Obukhov [3], and the review of d'Erceville and Kuznetz's [4] solution for inclined uniaxial anisotropic conductivity by Grubert [5], no significant attention has been given to the problem of exactly solving for the surface impedance anomalies of laterally inhomogeneous conductive media that have inclined conductivity anisotropy. In practical geophysical problems related to surface impedance interpretation, inclined uniaxial anisotropy occurs in structures with arbitrary shape for which an analytical solution may not be obtained easily, if at all [6]. For such problems, it is

essential to employ numerical methods of solution. Finite-element, finite-difference, and integral equation solutions have all been employed in modeling arbitrary two- (2-D) and three-dimensional (3-D) geological structures [7]. However, in these instances, the inhomogeneous models were considered as a spatial juxtaposition of different media with isotropic conductivity.

It has been recently demonstrated that 2-D problems involving TM-type incidence on media with inclined uniaxial anisotropic conductivity are equivalent to problems involving TM-type incidence on media with fundamental biaxial anisotropic conductivity [8]. The effective horizontal and vertical conductivity values are obtained from the diagonal components of the Euler rotation of the resistivity tensor into the horizontal and vertical planes, respectively, and were shown to be functions of the diagonal and nondiagonal terms from the Euler rotation of the conductivity tensor into the horizontal and vertical planes, respectively. In this letter, we use the self-consistent impedance method of Thiel and Mittra [9], [13], based on a finite-difference approximation of an *RC* circuit analogy, to demonstrate this principle of modeling 2-D inhomogeneous media with inclined uniaxial anisotropic conductivity in two dimensions.

II. FORMULATION

The numerical formulation is introduced using the Helmholtz equation for inclined uniaxial anisotropic media, and its properties for applications in the modeling of 2-D homogeneous TM-type wave propagation problems in inhomogeneous and anisotropic media. By considering the Maxwell equations excluding extraneous currents and sources in a coordinate system $\{x', y', z'\}$ inclined about the x axis by an angle α , one can obtain the Helmholtz equation for uniaxial anisotropic media

$$\frac{1}{\sigma_n} \frac{\partial^2 H_{x'}}{\partial y'^2} + \frac{1}{\sigma_t} \frac{\partial^2 H_{x'}}{\partial z'^2} - j\omega\mu H_{x'} = 0 \quad (1)$$

where σ_t is the conductivity parallel to the bedding plane; σ_n is the conductivity normal to the bedding plane; and both σ_t and σ_n can be complex. If we consider the x axis coordinate rotation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_m & \sin \alpha_m \\ 0 & -\sin \alpha_m & \cos \alpha_m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}(-\alpha) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2)$$

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The authors are with the School of Microelectronic Engineering, Griffith University—Nathan Campus, Brisbane, Queensland 4111, Australia (e-mail: glenn.wilson@griffith.edu.au; d.thiel@griffith.edu.au).

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such that the partial derivatives of any function f can be rewritten as

$$\frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial y^2} \cos^2 \alpha_m + 2 \frac{\partial^2 f}{\partial y \partial z} \sin \alpha_m \cdot \cos \alpha_m + \frac{\partial^2 f}{\partial z^2} \sin^2 \alpha_m \quad (3)$$

$$\frac{\partial^2 f}{\partial z'^2} = \frac{\partial^2 f}{\partial y^2} \sin^2 \alpha_m - 2 \frac{\partial^2 f}{\partial y \partial z} \sin \alpha_m \cdot \cos \alpha_m + \frac{\partial^2 f}{\partial z^2} \cos^2 \alpha_m \quad (4)$$

then (1) can be rewritten in yz coordinates as

$$\left(\frac{\cos^2 \alpha}{\sigma_n} + \frac{\sin^2 \alpha}{\sigma_t} \right) \frac{\partial^2 H_x}{\partial y^2} + \left(\frac{\sin^2 \alpha}{\sigma_n} + \frac{\cos^2 \alpha}{\sigma_t} \right) \frac{\partial^2 H_x}{\partial z^2} + 2 \left(\frac{1}{\sigma_n} - \frac{1}{\sigma_t} \right) \sin \alpha \cos \alpha \frac{\partial^2 H_x}{\partial y \partial z} - j\omega\mu H_x = 0. \quad (5)$$

From (5), the term containing the mixed partial derivative

$$2 \left(\frac{1}{\sigma_n} - \frac{1}{\sigma_t} \right) \sin \alpha \cos \alpha \frac{\partial^2 H_x}{\partial y \partial z}$$

is commonly called the shearing term and has been shown to vanish for laterally homogeneous and inhomogeneous media [8]. Equation (5) then becomes the Helmholtz equation for a biaxial medium in fundamental coordinates

$$\rho_{yy} \frac{\partial^2 H_x}{\partial y^2} + \rho_{zz} \frac{\partial^2 H_x}{\partial z^2} - j\omega\mu H_x = 0 \quad (6)$$

where the resistivity elements shown in (6) are the corresponding diagonal elements of the resistivity tensor of a uniaxial anisotropic medium rotated about the x axis, $\hat{\rho} = \mathbf{R}(\alpha)\hat{\rho}'\mathbf{R}^T(\alpha)$

$$\rho_{yy} = \frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} = \left(\sigma_{yy} - \frac{\sigma_{yz}\sigma_{zy}}{\sigma_{zz}} \right)^{-1} \quad (7)$$

$$\rho_{zz} = \frac{\sin^2 \alpha}{\sigma_t} + \frac{\cos^2 \alpha}{\sigma_n} = \left(\sigma_{zz} - \frac{\sigma_{zy}\sigma_{yz}}{\sigma_{yy}} \right)^{-1}. \quad (8)$$

The conductivity tensor elements in (7) and (8) are derived from the Euler rotation of the conductivity tensor of a uniaxial anisotropic medium rotated about the x axis, $\hat{\sigma} = \mathbf{R}(\alpha)\hat{\sigma}'\mathbf{R}^T(\alpha)$. To model 2-D media with inclined anisotropy, previous authors have developed numerical solutions to (5) [10]. In the following section, it is demonstrated with two model studies that a numerical solution of (6) obtains results nearly, if not, identical to the corresponding analytical solutions and results obtained by (5).

A 2-D self-consistent form of the impedance method was recently published, and its applications to surface impedance modeling of isotropic media were discussed [8]. The definition of self-consistent is that the magnetic field is assumed to be unknown everywhere in the solution space (with the exception of the source terms) and is independent of the model. Previous formulations of the impedance method [11] assumed that the magnetic field everywhere was known and was dependent on the

model. The basis of the self-consistent impedance method is the finite-difference solution to the Helmholtz (6)

$$\left(\frac{2}{(k_{z,i,k}\Delta z_{i,k})^2} + \frac{2}{(k_{y,i,k}\Delta y_{i,k})^2} + 1 \right) H_{x,i,k} - \frac{H_{x,i,k-1}}{(k_{z,i,k}\Delta z_{i,k})^2} - \frac{H_{x,i+1,k}}{(k_{y,i,k}\Delta y_{i,k})^2} - \frac{H_{x,i,k+1}}{(k_{z,i,k}\Delta z_{i,k})^2} - \frac{H_{x,i-1,k}}{(k_{y,i,k}\Delta y_{i,k})^2} = 0 \quad (9)$$

where $k_{y,i,k}$ and $k_{z,i,k}$ are the propagation coefficients in the y and z directions, respectively, written as

$$k_{y,i,k} = \sqrt{\frac{j\omega\mu_0}{\rho_{yy,i,k}}} \quad (10)$$

$$k_{z,i,k} = \sqrt{\frac{j\omega\mu_0}{\rho_{zz,i,k}}}. \quad (11)$$

The wave numbers are chosen such that $\text{Re } k_{(y,z),i,k} > 0$ to prevent exponentially divergent solutions. For a solution space consisting of N cells, from (9), one can then write the matrix equation

$$\mathbf{S}\mathbf{H} = \mathbf{J}_0 \quad (12)$$

where \mathbf{H} is a $1 \times N$ matrix of unknown magnetic field elements in the solution space, and \mathbf{J}_0 is the $1 \times N$ matrix of applied current densities (i.e., source terms) that must have nonzero terms to prevent nontrivial solutions in \mathbf{H} . For example, in homogeneous plane wave incidence, the impressed magnetic field is generated from a series of applied current densities in the x direction. \mathbf{S} is a sparse square matrix N^2 in size and is called the propagation matrix [9], [13]. The unknown magnetic field values are solved with the matrix equation

$$\mathbf{H} = \mathbf{S}^{-1}\mathbf{J}_0 \quad (13)$$

where \mathbf{S}^{-1} can be solved by any number of matrix inversion algorithms. The code developed for this letter was written in Matlab, and solves for the matrix inversion using a standard LU decomposition method. Dirichlet boundary conditions are used to define the magnetic field as the incident homogeneous plane wave. Neumann boundary conditions are employed to terminate the boundaries in the other directions. This is equivalent to using a perfect electric conductor to terminate the basement of the solution space. The surface impedance Z_{yx} at the earth-air interface, defined at the top of the (i, k) th cell, is written as

$$Z_{yx} = \frac{E_y}{H_x} = \frac{(H_{x,i,k} - H_{x,i,k-1})\rho_{yy,i,k}}{H_{x,i,k-1}\Delta z_{i,k}}. \quad (14)$$

It should be noted that the $H_{x,i,k-1}$ term in (14) is measured above the surface of the conducting half-space, and the $E_{y,i,k}$ term is measured at the surface of the conducting half-space. It is observed that having at least two rows of air cells above the surface of the conducting half-space provides accurate surface impedance results. Since the media above the half-space is perfectly insulating, H_x is approximately uniform over distances less than $\lambda/10$ above the surface. The basic field approximation, as given by (9), is formally identical to that resulting from a finite-difference approximation with a cell-centered TM-type magnetic field. This approach is not common in 2-D finite-difference modeling practice, but

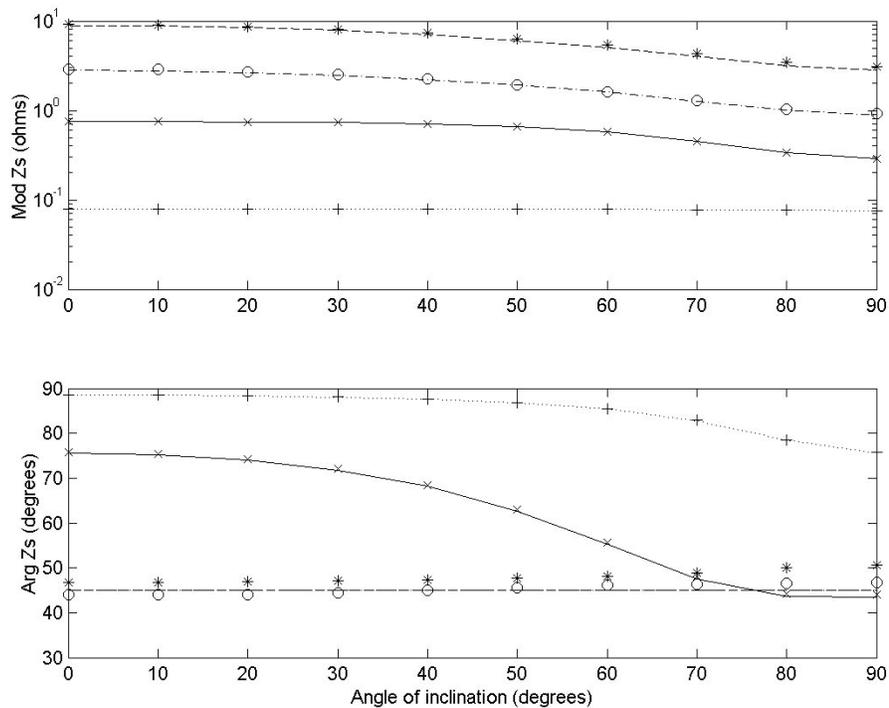


Fig. 1. Surface impedance magnitude and phase response for an anisotropic layer of 1000-m thickness with $\sigma_{t,1} = 0.001$ S/m and $\sigma_{n,1} = 0.01$ S/m above a perfectly conducting basement. The anisotropic conductivity tensor is rotated through the range $0 \leq \alpha \leq 90^\circ$. The impedance method solutions for 10 Hz (+), 100 Hz (x), 1 kHz (o), and 10 kHz (*) are presented. The analytical solutions for 10 Hz (dotted), 100 Hz (solid), 1 kHz (dashed-dotted), and 10 kHz (double-dashed) are also shown. Note that at 1 and 10 kHz, the surface impedance is equal to the intrinsic impedance of the layer, and the phase is equal to 45° .

standard 3-D staggered-grid modeling approaches [12] reduce to this case when homogeneity is assumed in one horizontal direction. Thus, based upon the approximation of (9), 2-D accuracy tests for staggered grid solutions can be made.

III. RESULTS

To demonstrate that an inclined uniaxial anisotropic conductor can be simulated as a fundamental biaxial anisotropic conductor, a number of cases involving inclined anisotropic conductivity are presented in which the surface impedance of the conducting half-space can be expressed as an analytical function to assess the accuracy of this formulation.

A. Homogeneous Layer Above a Perfect Electric Conductor

The analytical solution for the surface impedance of a TM-type wave normally incident upon a horizontally layered half-space was discussed in [2]. For a homogeneous layer with intrinsic impedance

$$Z_1 = \sqrt{j\omega\mu \left(\frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right)^{-1}} \quad (15)$$

terminated at $z = h$ by a perfect electric conductor, the surface impedance at $z = 0$ is given by the transmission line analogy

$$Z_{1,s} = \sqrt{j\omega\mu \left(\frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right)^{-1}} \tanh(k_1 h) \quad (16)$$

where

$$k_1 = \sqrt{j\omega\mu \left(\frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right)} \quad (17)$$

and $k_1 > 0$ to prevent an exponentially divergent solution in $Z_{1,s}$. A comparison between the surface impedance obtained from (16) and the self-consistent impedance method is shown in Fig. 1 for frequencies in the range $10 \text{ Hz} \leq f \leq 10 \text{ kHz}$ and is presented as a function of the angle of inclination of the anisotropic conductivity $0 \leq \alpha \leq 90^\circ$. The anisotropic layer was assumed to have $\sigma_{t,1} = 0.001$ S/m and $\sigma_{n,1} = 0.01$ S/m. The approximate model used 3×2 air cells, and 3×100 earth cells, and the depth between the air-half-space interface and the perfect electric conducting basement was constant at 1000 m. The cell sizes selected were $\Delta y = 20$ m and $\Delta z = 10$ m. The surface impedance magnitude agrees very well with the corresponding analytical solutions for all frequencies considered. The surface impedance phase, however, is more accurate at lower frequencies than at high frequencies. This difference is suggested to be related to the cell size chosen. For the lower frequencies, the cell size was several orders of magnitude less than the wavelength in the conducting half-space. At the higher frequencies presented, the cell size was approximately in the same order of magnitude as the wavelength considered. Such discretization errors are introduced in all approximate techniques and are not, therefore, unique to the self-consistent impedance method.

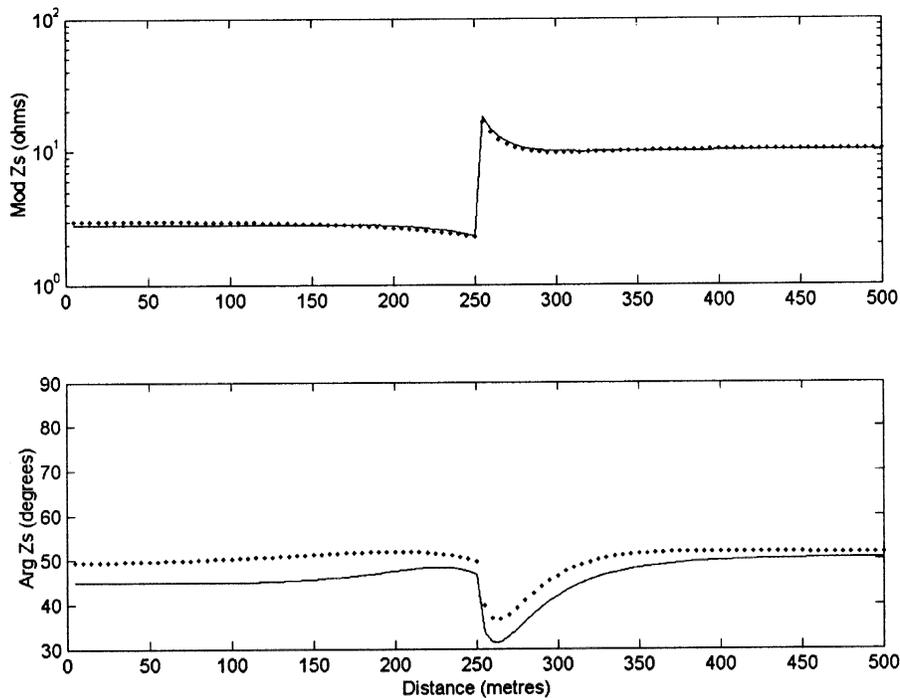


Fig. 2. Surface impedance magnitude and phase response at a single boundary at 10 kHz for a 500-m-wide dyke with $\sigma_{t,1} = 0.01$ S/m and $\sigma_{n,1} = 0.001$ S/m embedded in an otherwise homogeneous layer with $\sigma_{t,2} = 0.001$ S/m and $\sigma_{n,2} = 0.01$ S/m, terminated with a perfectly conducting basement. The depth of the inclusion and layer is 200 m. The impedance method response (dots) is shown above the analytical response (solid line).

B. Vertical Inclusion Embedded in a Homogeneous Layer Above a Perfect Electric Conductor

The method of analytically solving for the surface impedance of a homogeneous plane wave incident upon a vertical inclusion with inclined electrical anisotropy embedded in an otherwise homogeneous layer, above a perfect electric conducting basement, is discussed in [8]. It has been demonstrated that the inclined electrical anisotropy problem is equally posed as a fundamental electrical anisotropy problem. In Fig. 2, the surface impedance at 10 kHz is presented for an inclusion that is 500 m wide with $\sigma_{t,1} = 0.01$ S/m and $\sigma_{n,1} = 0.001$ S/m embedded in an otherwise homogeneous layer with $\sigma_{t,2} = 0.001$ S/m and $\sigma_{n,2} = 0.01$ S/m. The common depth of the dyke and homogeneous layer to the perfect electrically conducting basement is 200 m. The cell sizes selected were $\Delta y = 5$ m and $\Delta z = 5$ m, and the total solution space was 100×42 cells where two air cells were assigned in the model. There is good agreement in the surface impedance magnitude between the impedance method and the corresponding analytical solution, but the surface impedance phase has some inaccuracies. In this model, Δz is equivalent to a 40° phase shift in the $y < 250$ m medium and a 13° phase shift in the $y > 250$ m medium. From (14), this introduces phase errors of approximately 4° and 1° in the $y < 250$ m medium and $y > 250$ m medium, respectively.

IV. CONCLUSION

In this letter, the principle of modeling the surface impedance of a homogeneous TM-type plane wave incident upon an inhomogeneous half-space with inclined uniaxial electrical anisotropy as an equivalent half-space with fundamental electrical biaxial anisotropy has been demonstrated. The

self-consistent impedance method has been shown to accurately model the surface impedance response of these 2-D induction problems over a broad range of frequencies. While the impedance method has been introduced for this modeling concept in this letter, the modeling principles introduced can be applied to any 2-D numerical method. The surface impedance magnitude can be accurately modeled using the impedance method, but the surface impedance phase is shown to be inaccurate for laterally inhomogeneous and anisotropic models. This inaccuracy is reduced by using smaller cell sizes [9], [13]. It is noted here that this letter presents the first comparison of any numerical technique with the corresponding analytical solution for a laterally inhomogeneous and anisotropic half-space.

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